0. Jordan - Gerstenhaber Theory

Let $M_n(F)$ be the algebra of $n \times n$ matrices $/F$,

$\mathcal{N}_n \subset M_n(F)$ - nilpotent cone,

$GL_n(F)$ - general linear gp.

For $u \in \mathcal{N}_n$ $\quad \mathcal{O}_u := \{ A u A^{-1} \mid A \in GL_n(F) \}$

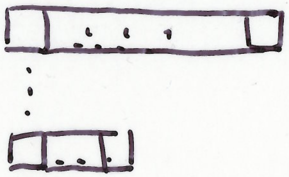
$\operatorname{Spec}(u) = \{0\}$

$$\mathcal{O}_u \longleftrightarrow \underset{\substack{\uparrow \\ \text{Jordan form}}}{J(u)} \longleftrightarrow \left\{ \begin{matrix} \text{block's} \\ \text{length} \end{matrix} \right\} \longleftrightarrow \lambda \vdash n$$

$$\mathcal{O}_u \longleftrightarrow \lambda = (\lambda_1, \ldots \lambda_k) \; ; \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k > 0$$

$$\sum_{j=1}^{k} \lambda_j = n$$

$\lambda \longleftrightarrow$



λ_1 boxes

λ_k - boxes

Young diagram

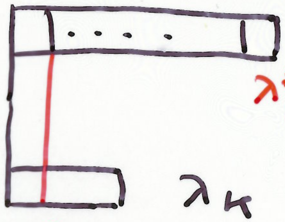
There is a bijection between nilpotent orbits in $\mathcal{N}_n$ and the set of Young diagrams with n boxes

$$\mathcal{O}_\lambda := \mathcal{O}_u \quad J(u) \leftrightarrow \lambda$$

Geometry of $\mathcal{O}_\lambda$ via combinatorics of Y. d.:

Jordan

1870-ties

$\mathcal{D}_\lambda =$  $\lambda_1 = m$ $\quad \lambda_k$

λ_m^*

$\lambda_1^* = k$

$\lambda = (\lambda_1, \dots, \lambda_k)$

$\lambda^* = (\lambda_1^*, \dots \lambda_m^*)$ - adjoin partition

= lengths of the columns of $\mathcal{D}_\lambda$

$u \in \mathcal{O}_\lambda$ $\quad$ $\operatorname{Rank} u^d = n - \sum_{i=1}^{d} \lambda_i^*$

$\dim \mathcal{O}_\lambda = n^2 - \sum_{j=1}^{m} (\lambda_j^*)^2$

Char $F = 0$

$\bar{\mathcal{O}}_\lambda = ?$

M. Gerstenhaber (1960-ties)

Dominance order on Y. d.: $\quad \lambda, \mu \vdash n$

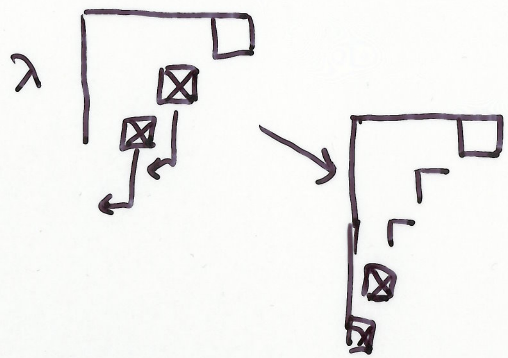
$\lambda = (\lambda_1, \dots, \lambda_k) \qquad \lambda_i \geq \lambda_{i+1} > 0$

$\mu = (\mu_1, \dots, \mu_l) \qquad \mu_i \geq \mu_{i+1} > 0$

Put $\lambda \geq \mu$ if $\sum_{i=1}^{d} \lambda_i \geq \sum_{i=1}^{d} \mu_i$ for any d:

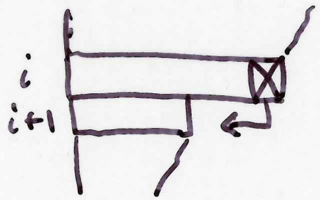
$1 \leq d \leq \min(k, l)$

$$\overline{\mathcal{O}_\lambda} = \coprod_{\mu \leq \lambda} \mathcal{O}_\mu$$



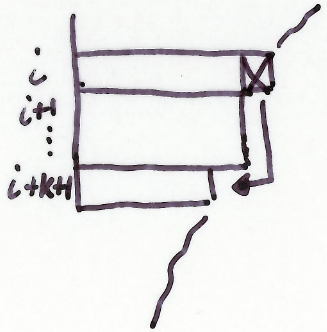
boxes "fall" down

In particular a cover of λ [that is $\mu < \lambda$ and $\mu \leq \nu \leq \lambda \Rightarrow \nu = \mu$ or $\nu = \lambda$]

(a) 

$$(\lambda_1, \dots, \lambda_i, \lambda_{i+1}, \dots)$$
$$\downarrow \quad \lambda_i \geq \lambda_{i+1} + 2$$
$$\mu = (\lambda_1, \dots, \lambda_i - 1, \lambda_{i+1} + 1, \dots)$$

$$\operatorname{codim}_{\overline{\mathcal{O}_\lambda}} \mathcal{O}_\mu = 2$$

(b) 

$$(\lambda_1, \dots, \lambda_i, \underbrace{\lambda_i - 1, \dots \lambda_i - 1}_{k \text{ times}}, \lambda_i - 2, \dots)$$
$$\downarrow$$
$$\mu = (\lambda_1, \dots, \underbrace{\lambda_i - 1, \lambda_i - 1, \dots, \lambda_i - 1}_{k+2 \text{ times}}, \dots) \quad \operatorname{codim}_{\overline{\mathcal{O}_\lambda}} \mathcal{O}_\mu = 2(k+1)$$

1. B-orbits in strictly upper-triangular matrices of nilpotent order 2.

Let $\mathfrak{n}_n \subset M_n(F)$ be the subalgebra of strictly upper-triangular matr.,

B_n - (Borel) subgp of $GL_n(F)$ of upper-triangular invertible matrices

$\mathfrak{n}_n / B_n$: $u \in \mathfrak{n}_n$ $\mathcal{B}_u = \{ A u A^{-1} \mid A \in B_n \}$

Canonical form in the spirit of Jordan?

Tragic news: for F, char $F = 0$ and $n \geq 6$ there is infinite number of B-orbits

Good news: $\mathcal{X}_n^2 = \{ u \in \mathfrak{n}_n : u^2 = 0 \}$

$S_n^2 = \{ \sigma \in S_n : \sigma^2 = Id \}$

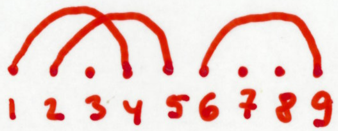
$$|\mathcal{X}_n^2 / B_n| = |S_n^2|$$

Combinatorial description in the spirit of J-G exists.

Link patterns. Example

$\sigma = (1,4)(2,5)(6,9) \in S_9^2$

$$M_\sigma = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix} \rightarrow \quad N_\sigma = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$P_\sigma =$ 

fixed pt's $\{3, 7, 8\}$

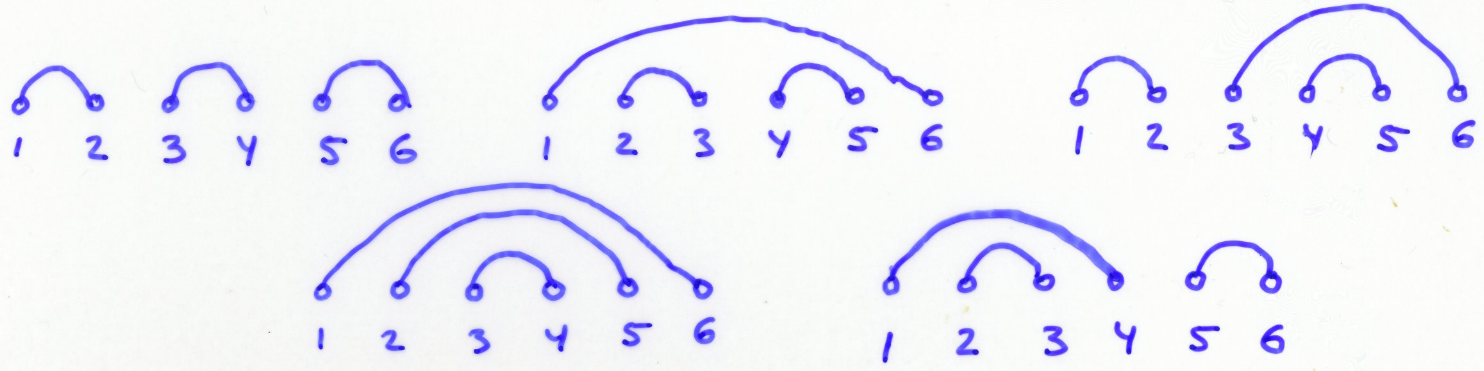
$\ell(\sigma) = 3$

$c(\sigma) = 1 \qquad f(\sigma) = 2 + 1 + 1 = 4$

$\dim \mathcal{B}_\sigma = 3 \cdot 6 - 1 - 4 = 13$

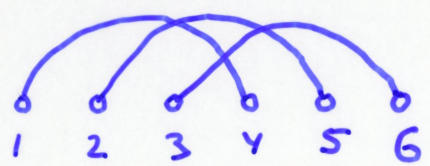
$$R_\sigma = \begin{pmatrix} \cdot & \cdot & \cdot & 1 & 2 & 2 & 2 & 2 & 3 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

<u>Example</u>: 5 "maximal" link patterns in $S_6^2(3)$: (dim $\mathcal{B}_\sigma = 3 \cdot 3 = 9$)

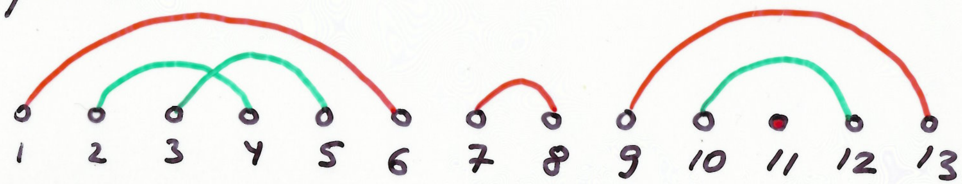


The only "minimal" link pattern in $S_6^2(3)$:

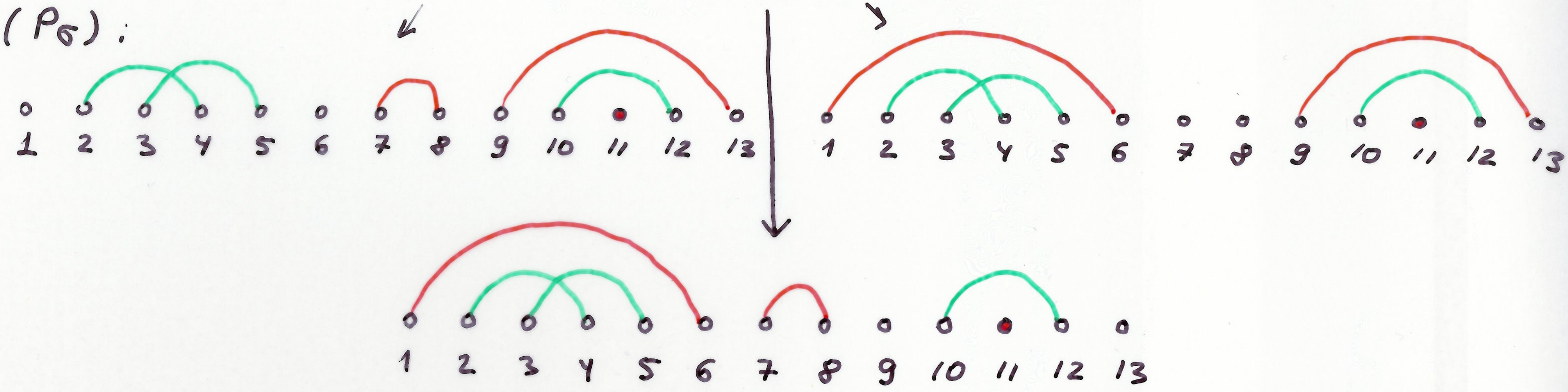
dim $\mathcal{B}_\sigma = 3 \cdot 3 - 3 = 6$



Example

$P_\sigma =$ 

$N(P_\sigma)$:

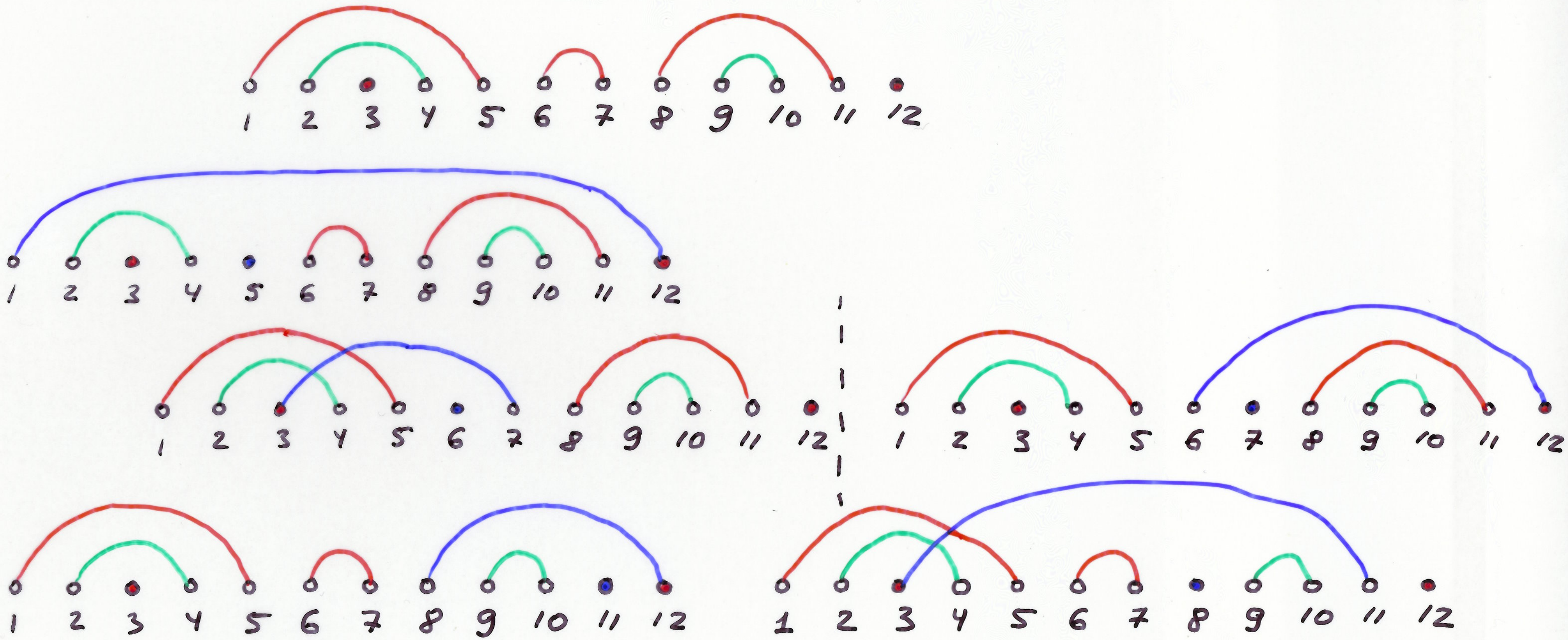


<u>Note</u>: In general $N(P_\sigma)$ is not equidimensional

However if $\sigma \in \mathfrak{S}_n^2(k)$ is of maximal possible dimension ($\dim \mathcal{B}_\sigma = k(n-k)$) then $N(P_\sigma)$ is equidimensional of $\operatorname{codim}_{\overline{\mathcal{B}}_\sigma} \mathcal{B}_\sigma{}' = n-2k+1$

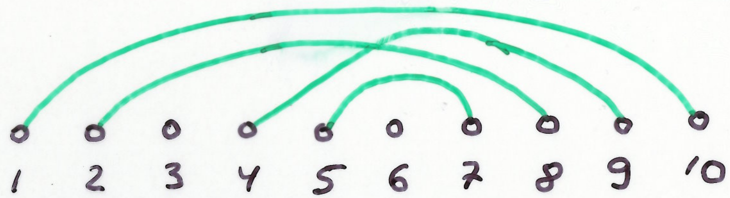
$\mathcal{D}(\sigma)$

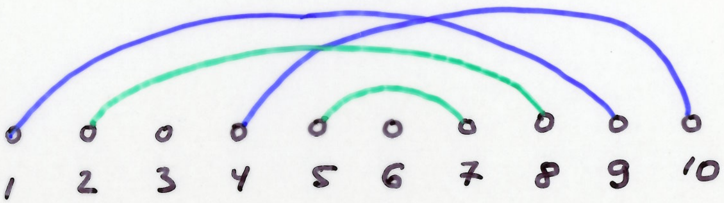
(a) Moving one of the end pt's of an external arc to the nearest fixed pt (on the same side)

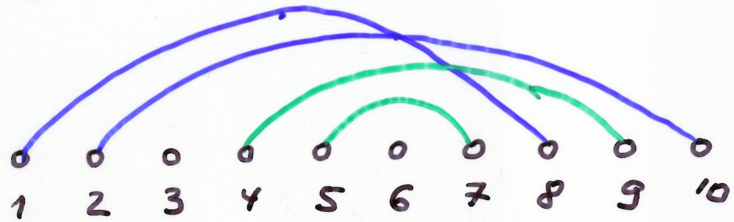


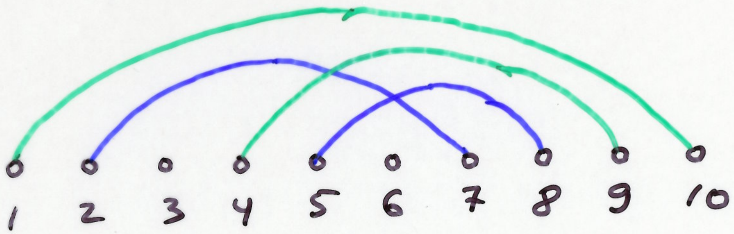
$\mathcal{D}(\sigma)$

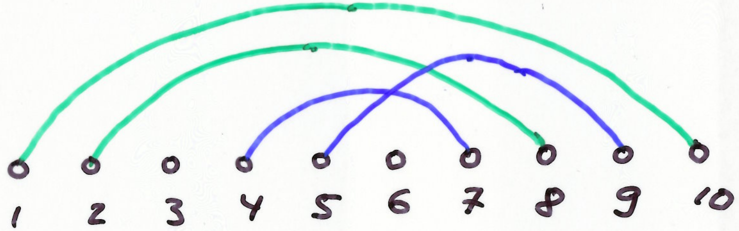
(b) Interchanging end pt's of concentric arcs with no concentric arcs in between:





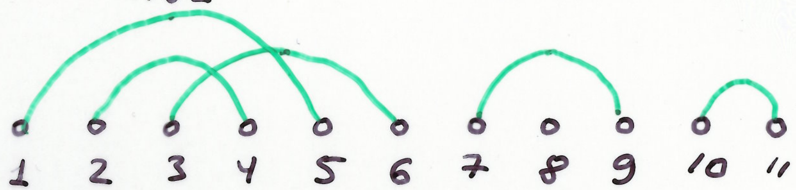


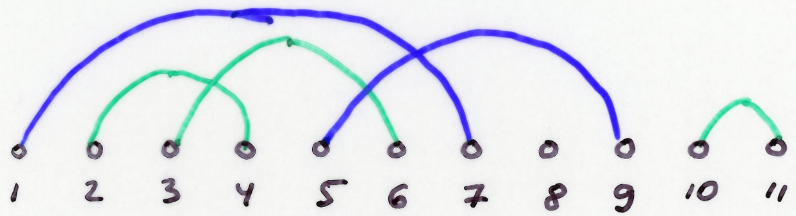


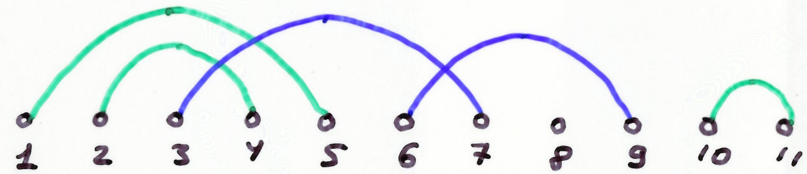


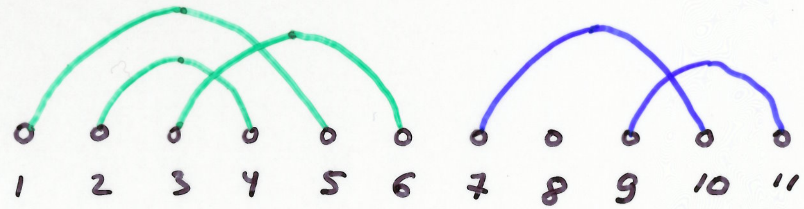
$\mathcal{D}(\sigma)$

(c) Let arc (i_2, d_2) be to the right of (i_1, d_1) $(d_1 < i_2)$ and such that there are no fixed points on $]d_1, i_2[$ in $\pi_{i_1,d_2}(\sigma)$. Then interchange d_1 and i_2









<u>Conclusion</u>: $\mathcal{D}(\sigma)$ is equidimensional of codim 1.

2. What are they good for?

1). Study of orbital varieties of nilpotent order 2

2). Intersections of the components of Springer fibers of n.o. 2

3). Study of spherical orbits (E. Smirnov)

Def. of o.v.: $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$; $\mathcal{O}$ - nilpotent orbit

The irreducible components of $\mathcal{O} \cap \mathfrak{n}$ are called orbital varieties associated to $\mathcal{O}$.

They are Lagrangian subvarieties of $\mathcal{O}$

(A. Joseph, N. Spaltenstein, R. Steinberg)

In $\mathfrak{g} = \mathfrak{sl}_n$ orbital varieties assoc. to $\mathcal{O}_\lambda$ are parametrized by Young tableaux of shape λ

Springer fiber:

Let $F = \{F_0 \subset F_1 \subset \ldots \subset F_n = \mathbb{C}^n\}$ be a full flag

Let $u \in \mathcal{N}$ be a nilpotent element

$\mathcal{F}_u = \{F : uF_i \subset F_{i-1} \quad \forall i \quad 1 \le i \le n\}$ is called a Springer fiber.

For $u \in \mathcal{O}_\lambda$ the irreducible components of $\mathcal{F}_u$ are equidimensional and parametrized by Young tableaux of shape λ. (N. Spaltenstein, J. A. Vargas, 1970-ties)

Given T, S of shape λ. Let $\mathcal{F}_T, \mathcal{F}_S$ be the corresponding components of $\mathcal{F}_u$.

$$\mathcal{F}_T \cap \mathcal{F}_S \quad ?$$

When are $\mathcal{F}_T \cap \mathcal{F}_S$ of codimension 1?

Are intersections reducible or not?

Connection to Kazhdan-Lusztig data?

1). $\lambda = (\lambda_1, 1, \dots, 1)$ - hook shape

$\mathcal{F}_T$ - nonsingular
The full description of $\mathcal{F}_T \cap \mathcal{F}_S$ was made by N. Spaltenstein and J. A. Vargas.
In particular $\mathcal{F}_T \cap \mathcal{F}_S$ are either irreducible or empty (70ties)

2). $\lambda = (\lambda_1, \lambda_2)$ - two row shape
$\mathcal{F}_T$ - nonsingular
The full description of $\mathcal{F}_T \cap \mathcal{F}_S$ was made by F. Y. C. Fung. In particular
$\mathcal{F}_T \cap \mathcal{F}_S$ are either irreducible or empty (2000)
Codimensions are computed with the help of meanders and they are given in terms of inner multiplications of Kazhdan-Lusztig basis of Temperley-Lieb algebras (computed by B.W. Westbury and J.J. Graham & G.I. Lehrer)

<u>Thm 1</u> (A. M. + G. N. J. Pagnon)
Given $u \in \mathcal{O}_\lambda$, let T, S be Y. t. of shape λ.
Let $\mathcal{V}_T, \mathcal{V}_S$ be orbital varieties associated to $\mathcal{O}_\lambda$
and $\mathcal{F}_T, \mathcal{F}_S$ components of $\mathcal{F}_u$.
Then the number of components of $\mathcal{V}_T \cap \mathcal{V}_S$ and
the codimensions of these components are equal
to the number of components of $\mathcal{F}_T \cap \mathcal{F}$ and
their codimensions.

<u>Thm 2</u> Each orbital variety of nilpotent order 2
has a dense B-orbit (it's link pattern is constructed
exactly as in T.-d. algebras)

Given $\sigma, \sigma' \in S_n^2$ put $(R_{\sigma,\sigma'})_{i,j} = \min((R_\sigma)_{i,j}, (R_{\sigma'})_{i,j})$

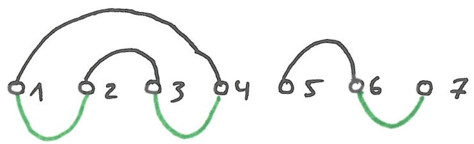
<u>Thm 3</u> (A. M. + G. N. J. Pagnon) For $\sigma, \sigma' \in S_n^2$

$$\overline{\mathcal{B}}_\sigma \cap \overline{\mathcal{B}}_{\sigma'} = \bigsqcup_{R_{\sigma''} \leq R_{\sigma,\sigma'}} \mathcal{B}_{\sigma''}$$

In particular $\overline{\mathcal{B}}_\sigma \cap \overline{\mathcal{B}}_{\sigma'}$ is irred. iff
there exists $\sigma'' \in S_n^2$ s.t. $R_{\sigma''} = R_{\sigma,\sigma'}$

Let $\sigma, \sigma' \in S_n^2(k)$ be maximal

$M_{\sigma,\sigma'}$ is a union of $P_\sigma, P_{\sigma'}$ one of them upward and the second downward:



A meander $M_{\sigma,\sigma'}$ is called even if any its connected path consists of even number of arcs and odd otherwise.

<u>Notes:</u> 1). $\mathcal{F}_T$ is singular in general for two-column Y. t.

2). For T, S - two column tableaux of the same shape $\mathcal{F}_T \cap \mathcal{F}_S \neq \emptyset$

3). In general $\mathcal{F}_T \cap \mathcal{F}_S$ is reducible and not of pure dimension

4). The combinatorics of meanders give the number of components in the intersection and their codim.

Thm 4 Given S, T Young tableaux of shape $(n-k,k)^*$ let $M_{S,T}$ be the corresponding meander.
Then $\dim(\mathcal{F}_T \cap \mathcal{F}_S) = \dim \mathcal{F}_T - 1$ iff $M_{S,T}$ is an even meander with $k-1$ cycles. In this case the intersection is irreducible.

Note: 1). Comparing our results to the result of Fung we get $\operatorname{codim}_{\mathcal{F}_T}(\mathcal{F}_T \cap \mathcal{F}_S) = 1$ iff $\operatorname{codim}_{\mathcal{F}_{T^t}}(\mathcal{F}_{T^t} \cap \mathcal{F}_{S^t}) = 1$

2). In other cases the intersections $(\mathcal{F}_{T^t} \cap \mathcal{F}_{S^t})$ are sometimes of higher codimensions and sometimes of lower codimensions.

3) In general $\bar{\mathcal{B}}_\sigma \cap \bar{\mathcal{B}}_{\sigma'} \cap \mathcal{O}_{N_\sigma}$ is reducible even if $\operatorname{codim}_{\bar{\mathcal{B}}_\sigma}(\bar{\mathcal{B}}_\sigma \cap \bar{\mathcal{B}}_{\sigma'} \cap \mathcal{O}_{N_\sigma}) = 1$ $\quad (\sigma, \sigma' \in S_n^2(k))$

Conjectures

1). $\operatorname{codim}_{\mathcal{F}_T} (\mathcal{F}_T \cap \mathcal{F}_S) = 1 \Longrightarrow \mathcal{F}_T \cap \mathcal{F}_S$ is irreducible

2) $\operatorname{codim}_{\mathcal{F}_T} (\mathcal{F}_T \cap \mathcal{F}_S) = 1$ iff

$\operatorname{codim}_{\mathcal{F}_{T^t}} (\mathcal{F}_{T^t} \cap \mathcal{F}_{S^t}) = 1$

3) Intersections of codim 1 are ruled by Kazhdan-Lusztig data ?