Existence and combinatorial model for Kirillov–Reshetikhin crystals

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References

This talk is based on the following papers:

- A. Schilling,
  Combinatorial structure of Kirillov–Reshetikhin crystals of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$,
  preprint arXiv:0704.2046[math.QA]
- M. Okado, A. Schilling,
  Existence of Kirillov–Reshetikhin crystals for nonexceptional types,
  preprint arXiv:0706.2224[math.QA]

Quantum algebras

Drinfeld and Jimbo $\sim$ 1984:
independently introduced quantum groups $U_q(\mathfrak{g})$

Kashiwara $\sim$ 1990:
crystal bases, bases for $U_q(\mathfrak{g})$-modules as $q \longrightarrow 0$
combinatorial approach

Lusztig $\sim$ 1990:
canonical bases
geometric approach

Applications in...

representation theory
$\rightsquigarrow$ tensor product decomposition
solvable lattice models
$\rightsquigarrow$ one point functions
conformal field theory
$\rightsquigarrow$ characters
number theory
$\rightsquigarrow$ modular forms
Bethe Ansatz
$\rightsquigarrow$ fermionic formulas
combinatorics
$\rightsquigarrow$ tableaux combinatorics
topological invariant theory
$\rightsquigarrow$ knots and links

Motivation

- Crystal bases are combinatorial bases for $U_q(\mathfrak{g})$ as $q \longrightarrow 0$
- Affine finite crystals:
  - appear in 1d sums of exactly solvable lattice models
  - path realization of integrable highest weight $U_q(\mathfrak{g})$-modules
  - fermionic formulas
- Irreducible finite-dimensional $U_q(\mathfrak{g})$-modules classified by Chari-Pressley via Drinfeld polynomials

Motivation

- Kirillov-Reshetikhin modules $W_s^{(r)}$ form special subset
  Conjecture [HKOTY]
  $W_s^{(r)}$ has a crystal basis $B^{r,s}$

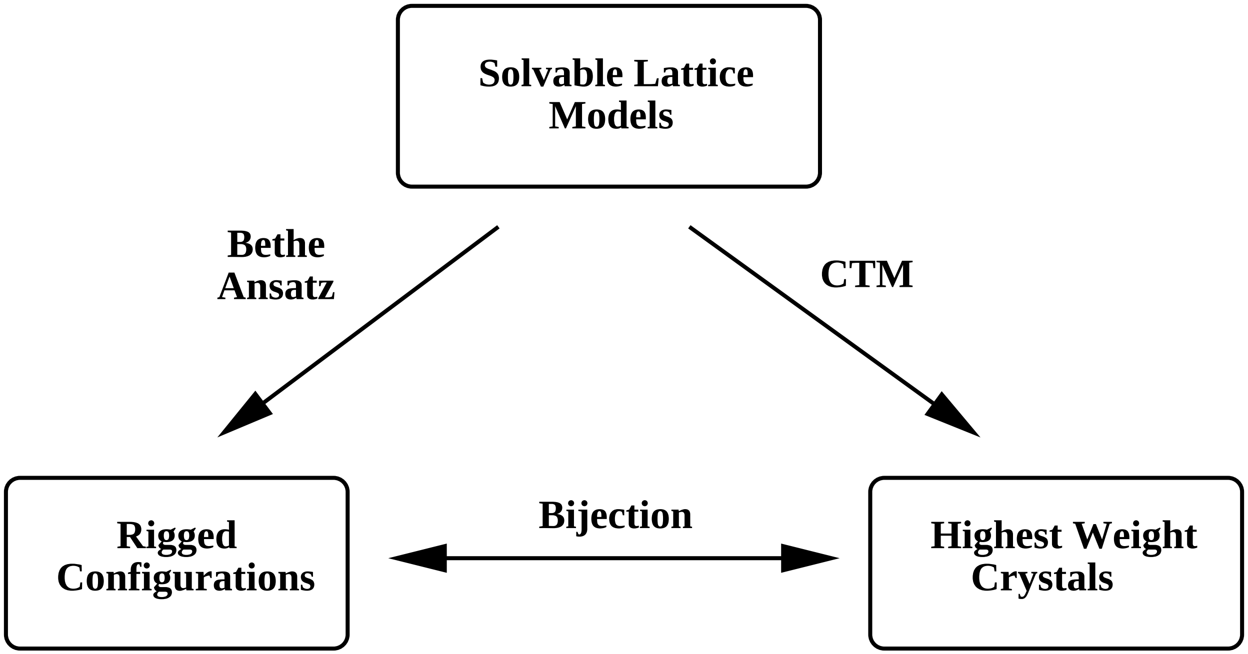
Motivation

- Kirillov-Reshetikhin modules $W_s^{(r)}$ form special subset
  Conjecture [HKOTY]
  $W_s^{(r)}$ has a crystal basis $B^{r,s}$

AIM:

- prove this conjecture for $\mathfrak{g}$ of nonexceptional type
- provide a combinatorial crystal $\tilde{B}^{r,s}$ for types $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$
- prove that $B^{r,s} \cong \tilde{B}^{r,s}$

Motivation



1988 Identity for Kostka polynomials Kerov, Kirillov, Reshetikhin

2001 $X = M$ conjecture of HKOTTY

Outline

I. Motivation

II. Existence of KR crystals $B^{r,s}$ for nonexceptional types

- Definition of KR modules
- Criterion for existence

III. Combinatorial KR crystals $\tilde{B}^{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

- Dynkin diagram automorphisms
- Classical crystal structure
- Affine crystal structure

IV. MuPAD-Combinat implementation

V. Outlook and open problems

II. Existence of KR crystals $B^{r,s}$ for nonexceptional types

Quantum affine algebras

$\mathfrak{g}$ symmetrizable affine Kac–Moody algebra

$U_q(\mathfrak{g})$ quantum affine algebra associated to $\mathfrak{g}$:
associative algebra over $\mathbb{Q}(q)$ with 1 generated by
e_i, f_i, q^h for $i \in I$, $h \in P^*$

$\{\alpha_i\}_{i\in I}$ simple roots, $\{h_i\}_{i\in I}$ simple coroots
c canonical central element, δ generator of null roots
$P = \bigoplus_i \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$ weight lattice

A subring of $\mathbb{Q}(q)$ of rational functions without poles at $q = 0$

$$A_{\mathbb{Z}} = \{f(q)/g(q) \mid f(q), g(q) \in \mathbb{Z}[q], g(0) = 1\}$$

$$K_{\mathbb{Z}} = A_{\mathbb{Z}}[q^{-1}]$$

Prepolarization

Let M be a $U_q(\mathfrak{g})$-module.
A symmetric bilinear form $(\,,\,) : M \otimes_{\mathbb{Q}(q)} M \to \mathbb{Q}(q)$ is called prepolarization if

$$(q^h u, v) = (u, q^h v)$$
$$(e_i u, v) = (u, q_i^{-1} t_i^{-1} f_i v)$$
$$(f_i u, v) = (u, q_i^{-1} t_i e_i v)$$

with $q_i = q^{(\alpha_i,\alpha_i)/2}, t_i = q_i^{h_i}$.
A prepolarization is called polarization if it is positive definite using the order

$$f > g \quad \text{iff} \quad f - g \in \cup_{n\in\mathbb{Z}}\{q^n(a + qA) \mid a > 0\}$$

Criterion for existence

M finite-dimensional integrable $U'_q(\mathfrak{g})$-module
$(\,,\,)$ prepolarization on M
$M_{K_\mathbb{Z}}$ submodule of M such that $(M_{K_\mathbb{Z}}, M_{K_\mathbb{Z}}) \subset K_\mathbb{Z}$
$\lambda_1, \ldots, \lambda_m \in \overline{P}_+$
Assumptions **A**:

1. $\dim M_{\lambda_k} \leq \sum_{j=1}^{m} \dim \overline{V}(\lambda_j)_{\lambda_k}$
2. There exist $u_j \in (M_{K_\mathbb{Z}})_{\lambda_j}$ such that

$$(u_j, u_k) \in \delta_{j,k} + qA$$

$$(e_i u_j, e_i u_j) \in q q_i^{-2(1+\langle h_i, \lambda_j \rangle)} A$$

Criterion for existence

If Assumption **A** holds:

Theorem: [KMN^2]

(i) $(\,,\,)$ is a polarization on M

(ii) $M \cong \bigoplus_j \overline{V}(\lambda_j)$ as $U_q(\mathfrak{g}_0)$-modules

(iii) (L, B) is a crystal pseudobase of M, where

$$L = \{u \in M \mid (u, u) \in A\}$$
$$B = \{b \in M_{K_\mathbb{Z}} \cap L / M_{K_\mathbb{Z}} \cap qL \mid (b, b)_0 = 1\}$$

$(\,,\,)_0$ is $\mathbb{Q}$-valued symmetric bilinear form on L/qL induced by $(\,,\,)$.

KR modules

Chari-Pressley classified all irreducible finite-dimensional affine $U_q(\mathfrak{g})$-modules via Drinfeld polynomials.

KR modules $W_s^{(r)}$ ($s \in \mathbb{Z}_{>0}, r = 1, \ldots, n$) correspond to the Drinfeld polynomials

$$P_j(u) = \begin{cases} (1 - a_r q_r^{1-s}) \cdots (1 - a_r q_r^{s-1} u) & j = r \\ 1 & j \neq r \end{cases}$$

for some $a_r \in \mathbb{Q}(q)$

Construction of KR modules

$V(\lambda)$ extremal weight module
level 0 fundamental weight $\varpi_i = \Lambda_i - \langle c, \Lambda_i \rangle \Lambda_0$
Define $U'_q(\mathfrak{g})$-module $W(\varpi_i)$ as

$$W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i)$$

where z_i is a $U'_q(\mathfrak{g})$-module automorphism of $V(\varpi_i)$ of weight $d_i\delta$

$$u_{\varpi_i} \mapsto u_{\varpi_i + d_i\delta} \qquad d_i = \max\{1, (\alpha_i, \alpha_i)/2\}$$

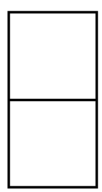
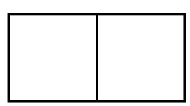
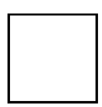
$W_s^{(r)}$ can be obtained by from $W(\varpi_r)$ by the fusion construction

Existence

Theorem[Okado,S.]
$W_s^{(r)}$ has a crystal basis $B^{r,s}$.

Assumption 1. follows from recent work by Nakajima and Hernandez on characters of KR-modules

Assumption 2. follows by finding appropriate λ_j and explicitly calculating the prepolarization in the cases

- Case : $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$
- Case : $C_n^{(1)}$
- Case : $A_{2n}^{(2)}$, $D_{n+1}^{(2)}$

Existence

Theorem[Okado,S.]
$W_s^{(r)}$ has a crystal basis $B^{r,s}$.

Remark: [KMN^2] proved the existence of $B^{r,s}$ for type $A_n^{(1)}$ and for other types for special r, s.

III. Combinatorial KR crystals $\tilde{B}^{r,s}$ of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

Axiomatic Crystals

A $U_q(\mathfrak{g})$-crystal is a nonempty set B with maps

$$\mathrm{wt}\colon B \to P$$

$$e_i, f_i\colon B \to B \cup \{\emptyset\} \quad \text{for all } i \in I$$

satisfying

$$f_i(b) = b' \Leftrightarrow e_i(b') = b \qquad \text{if } b, b' \in B$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \qquad \text{if } f_i(b) \in B$$

$$\langle h_i\, , \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b)$$

Write $b \xrightarrow{\;i\;} b'$ for $b' = f_i(b)$

KR crystals

$\mathfrak{g}$ affine Kac–Moody algebra

$W_s^{(r)}$ KR module indexed by $r \in \{1, \ldots, n\}$, $s \geq 1$

$\rightsquigarrow$ finite-dimensional $U_q'(\mathfrak{g})$-module

Chari proved

$$W_s^{(r)} \cong \bigoplus_{\Lambda} \overline{V}(\Lambda) \quad \text{as } U_q(\mathfrak{g}_0)\text{-module}$$

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$\mathfrak{g}$ of type $A_n^{(1)} \Rightarrow \mathfrak{g}_0$ of type A_n

$$W_s^{(r)} \cong \overline{V}\left(\left. \underbrace{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \end{array}}_{s} \right\} r \right)$$

KR crystals

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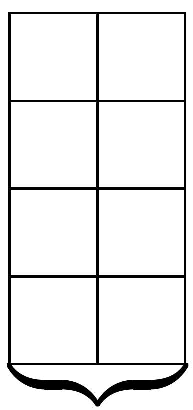
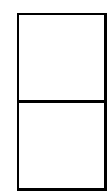
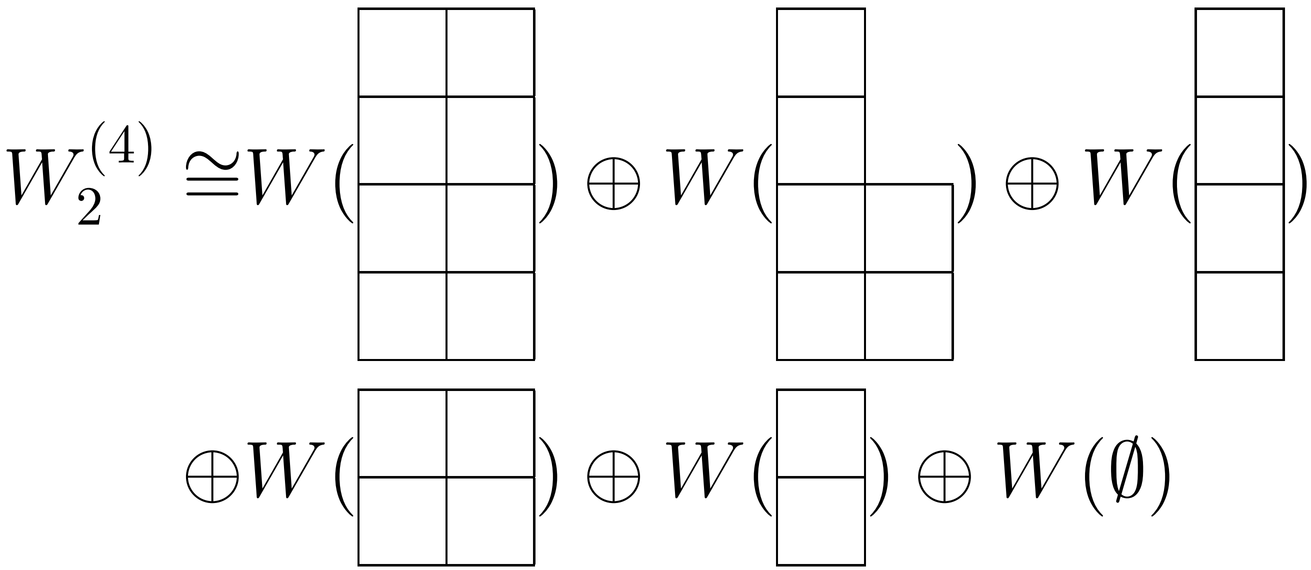
$\mathfrak{g}$ of type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)} \Rightarrow \mathfrak{g}_0$ of type D_n, B_n, C_n

sum over  r with vertical dominos  removed

s

Example

Type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

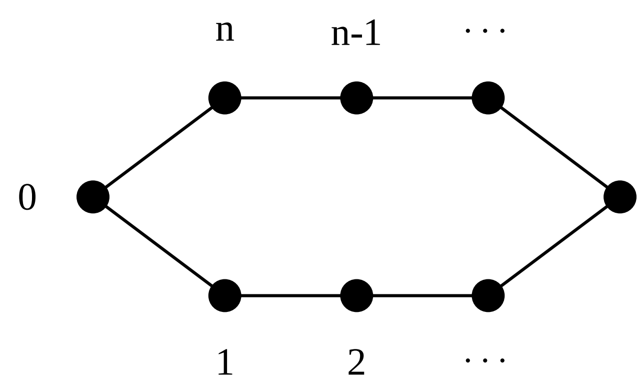


Dynkin automorphism

Type $A_n^{(1)}$:

KMN[2] proved existence of crystals $B^{r,s}$ for $W^{r,s}$
Shimozono gave the combinatorial structure of $B^{r,s}$ using σ

$A_n^{(1)}$



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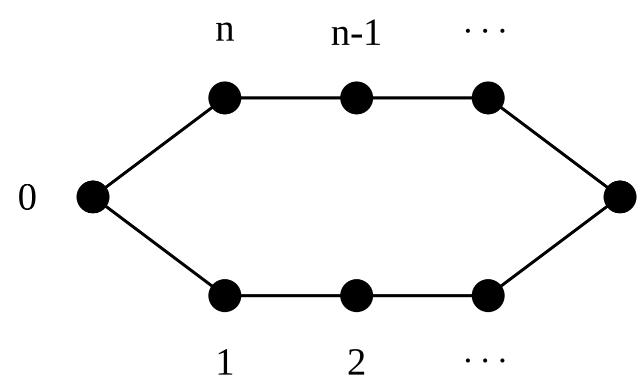


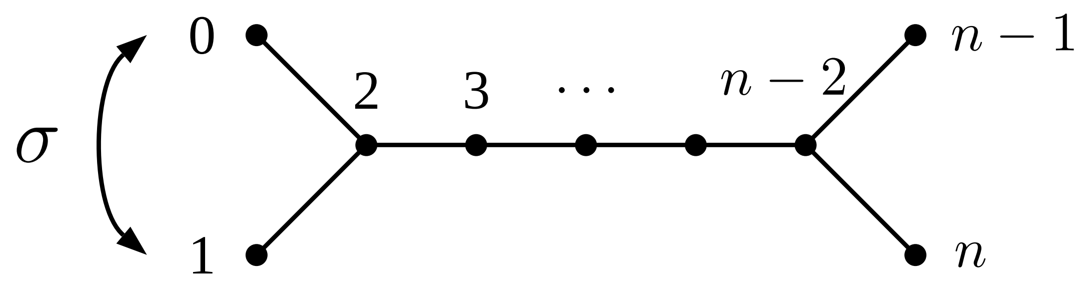
$$e_0 = \sigma^{-1} \circ e_1 \circ \sigma$$

$$f_0 = \sigma^{-1} \circ f_1 \circ \sigma$$

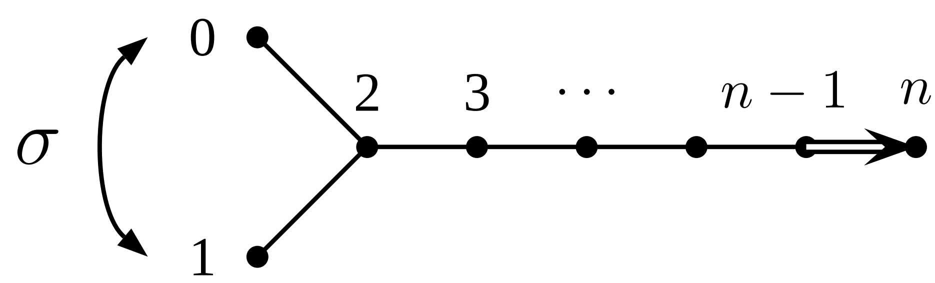
Dynkin automorphism

Type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$:

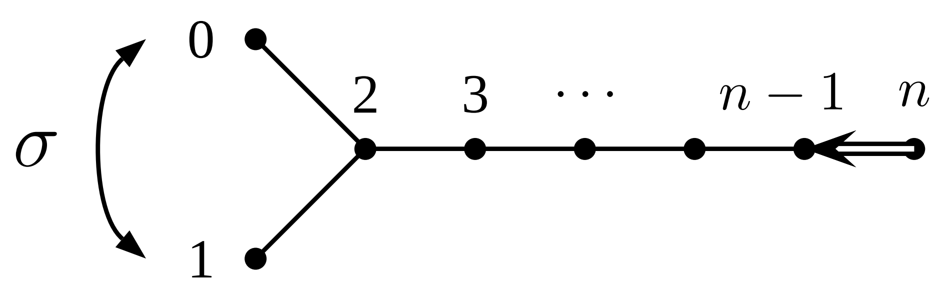
Type $D_n^{(1)}$:



Type $B_n^{(1)}$:

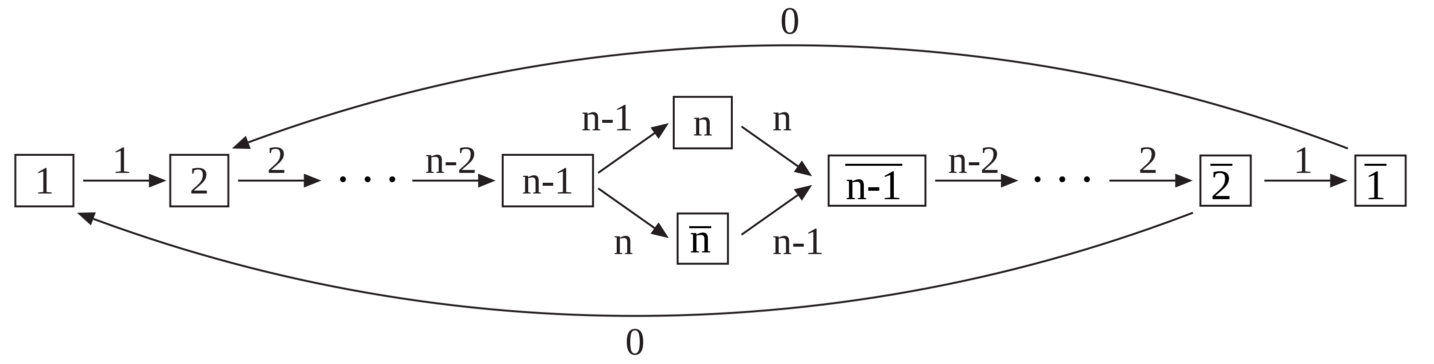
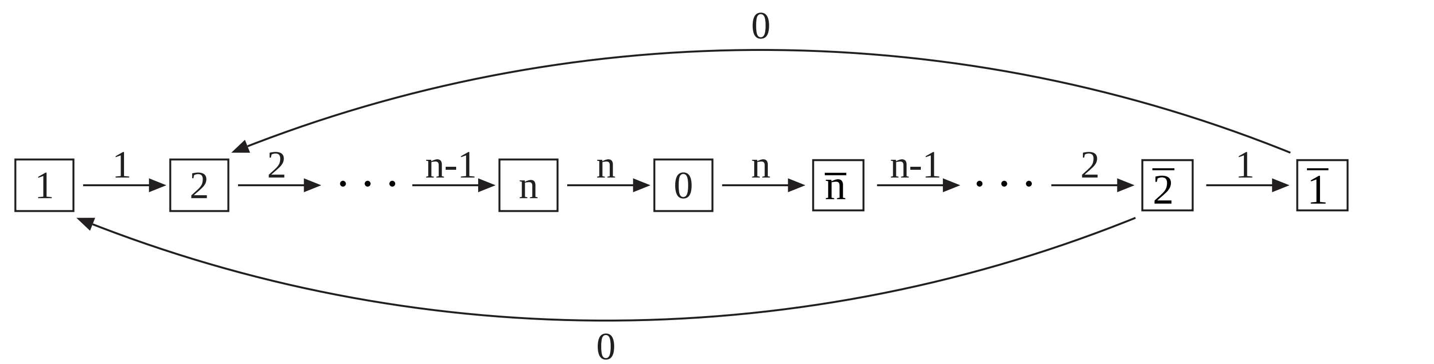
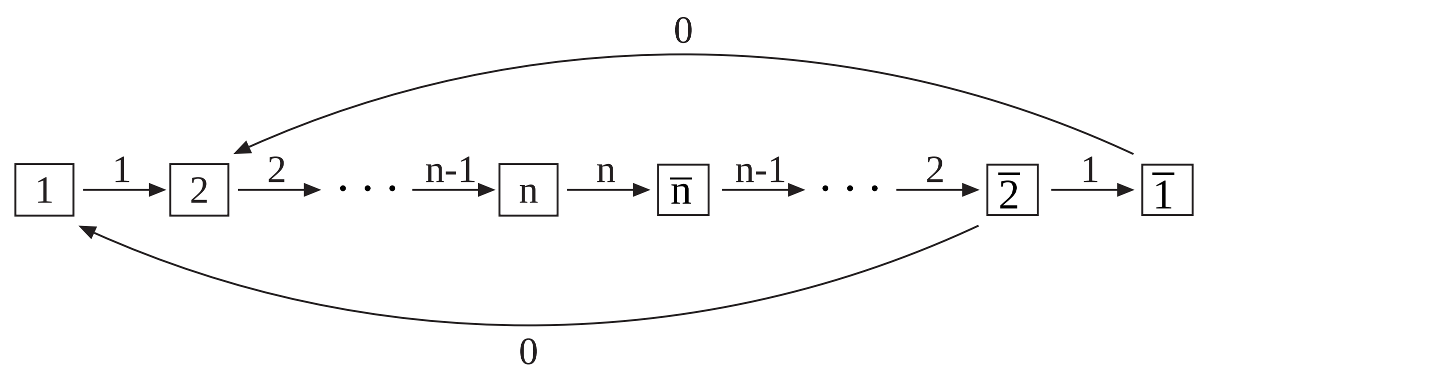


Type $A_{2n-1}^{(2)}$:



$e_0 = \sigma \circ e_1 \circ \sigma$ and $f_0 = \sigma \circ f_1 \circ \sigma$

Crystals $B^{1,1}$

$D_n^{(1)}$	
$B_n^{(1)}$	
$A_{2n-1}^{(2)}$	

Classical decomposition

By construction

$$B^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda)$$

as $X_n = D_n, B_n, C_n$ crystals

$\Rightarrow$ crystal arrows f_i, e_i are fixed for $i = 1, 2, \ldots, n$

Classical crystal

$$B(\Lambda) \subset B(\square)^{\otimes|\Lambda|}$$

highest weight

$$\begin{array}{|c|c|c|}\hline 4 \\ \cline{1-1} 3 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}$$

$$\mapsto \boxed{4} \otimes \boxed{3} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1}$$

f_i, e_i for $i = 1, 2, \ldots, n$ act by tensor product rule

$$b \quad \otimes \quad b'$$

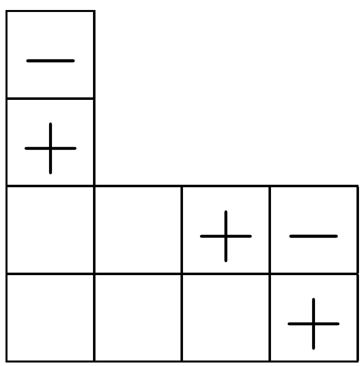
$$\underbrace{- - -}_{\varphi_i(b)}\underbrace{+ + +}_{\varepsilon_i(b)} \qquad \underbrace{- -}_{\varphi_i(b')}\underbrace{+ + + +}_{\varepsilon_i(b')}$$

Definition of σ

$X_n \to X_{n-1}$ branching

$$B_{X_n}(\Lambda) \cong \bigoplus_{\substack{\pm \text{ diagrams } P \\ \text{outer}(P) = \Lambda}} B_{X_{n-1}}(\text{inner}(P))$$

$\pm$ diagrams



$\lambda \subset \mu \subset \Lambda$

inner shape outer shape

Λ/μ horizontal strip filled with $-$

μ/λ horizontal strip filled with $+$

Definition of σ

X_{n-1} highest weight vectors
are in bijection with $\pm$ diagrams via Φ

Φ :

$-$			
$+$			
		$+$	$-$
			$+$

$\mapsto$

$\bar{4}$			
4			
2	3	3	$\bar{1}$
1	1	2	2

Definition of σ

X_{n-1} highest weight vectors
are in bijection with $\pm$ diagrams via Φ

$$\Phi : \begin{array}{|c|c|c|c|} \hline - & & & \\ \hline + & & & \\ \hline & & + & - \\ \hline & & & + \\ \hline \end{array} \quad \mapsto \quad \begin{array}{|c|c|c|c|} \hline \bar{4} & & & \\ \hline 4 & & & \\ \hline 2 & 3 & 3 & \bar{1} \\ \hline 1 & 1 & 2 & 2 \\ \hline \end{array}$$

$$\vec{a} = (1, 2, \quad 1, 2, 3, 4, 5, 6, 4, \quad 1, 2, 3, 4, 5, 6, 4, 3, 2)$$

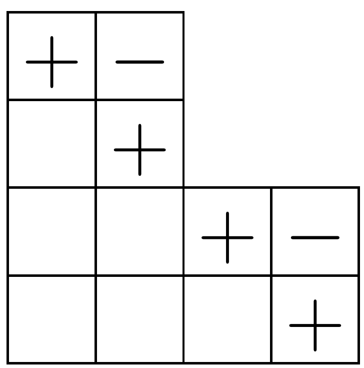
$$\Phi(P) = f_{\vec{a}} \quad \begin{array}{|c|c|c|c|} \hline 4 & & & \\ \hline 3 & & & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline \end{array}$$

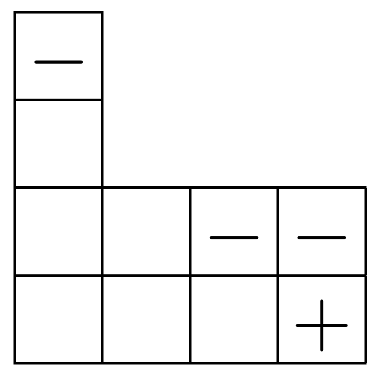
Definition of σ

σ on $\pm$ diagrams

P $\pm$ diagram of shape Λ/λ
columns of height h in λ

$$\begin{aligned} h \not\equiv r \bmod 2: &\quad \text{interchange number of} \\ &\quad + \text{ and } - \text{ above } \lambda \\ h \equiv r \bmod 2: &\quad \text{interchange number of} \\ &\quad \mp \text{ and empty above } \lambda \end{aligned}$$

$P =$ 

$\mathfrak{S}(P) =$ 

$r \geq 6$
$s = 5$

Definition of σ

σ on tableaux

$b \in \tilde{B}^{r,s}$

$$e_{\overrightarrow{\mathbf{a}}} := e_{a_1} \cdots e_{a_\ell} \quad \text{such that } e_{\overrightarrow{\mathbf{a}}}(b) \text{ is } X_{n-1} \text{ highest weight vector}$$

$$f_{\overleftarrow{\mathbf{a}}} := f_{a_\ell} \cdots f_{a_1}$$

Then

$$\sigma(b) = f_{\overleftarrow{\mathbf{a}}} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\overrightarrow{\mathbf{a}}}(b)$$

Example

$\tilde{B}^{4,5}$ of type $D_6^{(1)}$

$$b = \begin{array}{|c|c|cc}\hline \bar{4} & \bar{4} & \multicolumn{2}{c}{} \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \xrightarrow{e_4 e_6 e_5 e_4 e_3 e_2 e_2} \begin{array}{|c|c|cc}\hline 4 & \bar{4} & \multicolumn{2}{c}{} \\ \hline 3 & 4 & \multicolumn{2}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\ \hline 1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

Example

$\tilde{B}^{4,5}$ of type $D_6^{(1)}$

$$
b = \begin{array}{|c|c|cc}
\hline
\bar{4} & \bar{4} & & \\
\hline
3 & 4 & & \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}
\xrightarrow{e_4 e_6 e_5 e_4 e_3 e_2 e_2}
\begin{array}{|c|c|cc}
\hline
4 & \bar{4} & & \\
\hline
3 & 4 & & \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
$$

$$
\xrightarrow{\Phi^{-1}}
\begin{array}{|c|c|cc}
\hline
+ & - & & \\
\hline
 & + & & \\
\hline
 & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\
\hline
 & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\
\hline
\end{array}
\xrightarrow{\mathfrak{S}}
\begin{array}{|c|ccc}
\hline
- & & & \\
\hline
 & & & \\
\hline
 & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} & \multicolumn{1}{c|}{-} \\
\hline
 & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{+} \\
\hline
\end{array}
$$

Example

$\tilde{B}^{4,5}$ of type $D_6^{(1)}$

$$
b = \begin{array}{|c|c|cc}
\hline
\bar{4} & \bar{4} \\
\hline
3 & 4 \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}
\xrightarrow{e_4 e_6 e_5 e_4 e_3 e_2 e_2}
\begin{array}{|c|c|cc}
\hline
4 & \bar{4} \\
\hline
3 & 4 \\
\hline
2 & 3 & \multicolumn{1}{|c|}{\bar{1}} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
$$

$$
\xrightarrow{\Phi^{-1}}
\begin{array}{|c|c|cc}
\hline
+ & - \\
\hline
 & + \\
\hline
 & & \multicolumn{1}{|c|}{-} & \multicolumn{1}{c|}{-} \\
\hline
 & & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} \\
\hline
\end{array}
\xrightarrow{\mathfrak{S}}
\begin{array}{|c|ccc}
\hline
- \\
\hline
 \\
\hline
 & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{+} & \multicolumn{1}{c|}{-} \\
\hline
 & \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{+} \\
\hline
\end{array}
$$

$$
\xrightarrow{\Phi}
\begin{array}{|c|ccc}
\hline
\bar{3} \\
\hline
4 \\
\hline
3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\
\hline
\end{array}
\xrightarrow{f_2 f_2 f_3 f_4 f_5 f_6 f_4}
\begin{array}{|c|ccc}
\hline
\bar{2} \\
\hline
\bar{4} \\
\hline
3 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{\bar{1}} \\
\hline
1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} \\
\hline
\end{array}
= \sigma(b)
$$

Definition of $\tilde{B}^{r,s}$

$\tilde{B}^{r,s}$ is the crystal with the classical decomposition

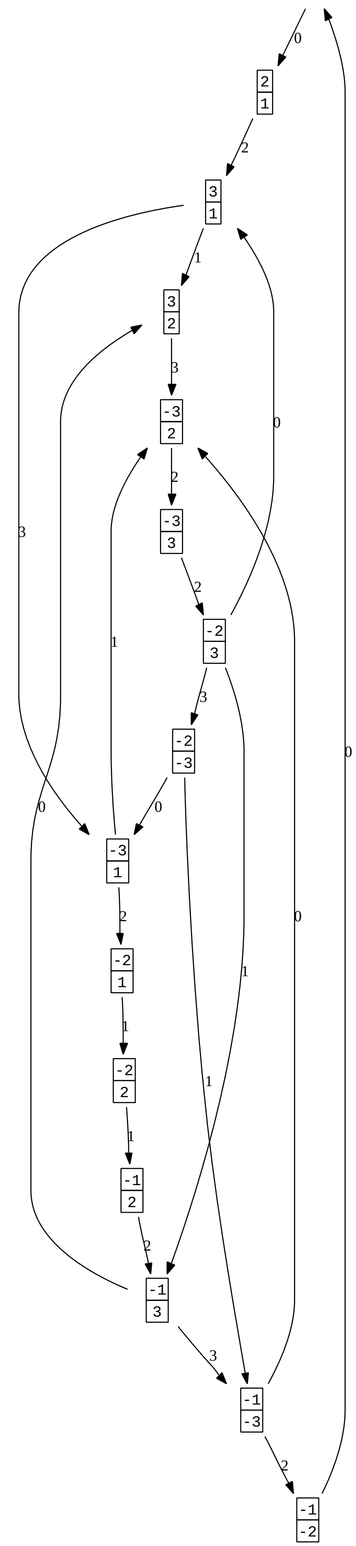
$$\tilde{B}^{r,s} \cong \bigoplus_{\Lambda} B(\Lambda) \quad \text{as } X_n = D_n, B_n, C_n \text{ crystals}$$

and

$$f_0 = \sigma \circ f_1 \circ \sigma$$
$$e_0 = \sigma \circ e_1 \circ \sigma$$

Example

$\tilde{B}^{2,1}$ type $A_5^{(2)}$



Uniqueness

B, B' I-crystals
$B \cong B'$ isomorphism of J-crystals where $J \subset I$

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Proposition. Suppose there exist two isomorphisms

$$\Psi_0 : \tilde{B}^{r,s} \cong B \quad \text{as } \{1, 2, \ldots, n\}\text{-crystals}$$
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Then $\Psi_0(b) = \Psi_1(b)$ for all $b \in \tilde{B}^{r,s}$ and hence there exists an I-crystal isomorphism

$$\Psi : \tilde{B}^{r,s} \cong B$$

Uniqueness

Theorem. [Okado, S.] For type $D_n^{(1)}$, $B_n^{(1)}$, $A_{2n-1}^{(2)}$

$$\tilde{B}^{r,s} \cong B^{r,s}$$

Proof. $\tilde{B}^{r,s}$ and $B^{r,s}$ have the same structure as

$\{1, 2, \ldots, n\}$-crystals (by construction)

$\{0, 2, \ldots, n\}$-crystals (by application of σ)

By previous Proposition there exists an isomorphism of I-crystals

$$\Psi : \tilde{B}^{r,s} \cong B^{r,s}$$

IV. MuPAD-Combinat implementation

MuPAD-Combinat...

... is an open source algebraic combinatorics package for the computer algebra system MuPAD [Hivert, Thiéry]

```
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

                                    +---+
                                    | 3 |
                                    +---+
                                    | 1 |
                                    +---+
>> t::e(0)

                                    +----+
                                    | -2 |
                                    +----+
                                    | 3  |
                                    +----+
```

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```
>> KR:=crystals::kirillovReshetikhin(2,2,["D",4,1]):
>> t:=KR([[3],[1]])

                                    +---+
                                    | 3 |
                                    +---+
                                    | 1 |
                                    +---+

>> t::sigma()

                                  +----+----+
                                  | -2 | -1 |
                                  +----+----+
                                  | 2  | 3  |
                                  +----+----+
```

Future

- Combinatorial structure for other KR crystals $C_n^{(1)}$, $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$,...
- $X = M$ conjecture for all types
- Level restriction