Lectures on Electric-Magnetic Duality and the Geometric Langlands Program

Anton Kapustin

California Institute of Technology, Pasadena, CA 91125, U.S.A.

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Abstract

These lecture notes are based on the master class given at the Center for the Topology and Quantization of Moduli Spaces, University of Aarhus, August 2007. I provide an introduction to the recent work on the Montonen-Olive duality of $\mathcal{N} = 4$ super-Yang-Mills theory and the Geometric Langlands Program.

1 Introduction

The Langlands Program is a far-reaching collection of theorems and conjectures about representations of the absolute Galois group of certain fields. For a recent accessible review see [1]. V. Drinfeld and G. Laumon [2] introduced a geometric analogue which deals with representations of the fundamental group of a Riemann surface C, or, more generally, with equivalence classes of homomorphisms from $\pi_1(C)$ to a reductive algebraic Lie group $G_{\mathbb{C}}$ (which we think of as a complexification of a compact reductive Lie group G). From the geometric viewpoint, such a homomorphism corresponds to a flat connection on a principal $G_{\mathbb{C}}$ bundle over C. The Geometric Langlands Duality associates to an irreducible flat $G_{\mathbb{C}}$ connection a certain D-module on the moduli stack of holomorphic ${}^LG$-bundles on C. Here ${}^LG$ is, in general, a different compact reductive Lie group called the Langlands dual of G. The group ${}^LG$ is defined by the condition that the lattice of homomorphisms from $U(1)$ to a maximal torus of G be isomorphic to the weight lattice of ${}^LG$. For example, the dual of $SU(N)$ is $SU(N)/\mathbb{Z}_N$, the dual of $Sp(N)$ is $SO(2N+1)$, while the groups $U(N), E_8, F_4$, and G_2 are self-dual.

The same notion of duality for Lie groups appeared in the work of P. Goddard, J. Nuyts and D. Olive on the classification of magnetic sources in gauge theory [3]. These authors found that magnetic sources in a gauge theory with gauge group G are classified by irreducible representations of the group ${}^LG$. On the basis of this, C. Montonen and D. Olive conjectured [4] that Yang-Mills theories with gauge groups G and ${}^LG$ might be isomorphic on the quantum level. This conjecture can be regarded as a generalization of the electric-magnetic duality in quantum Maxwell theory. Later H. Osborn [5] noticed that the Montonen-Olive conjecture is more likely to hold for $\mathcal{N}=4$ supersymmetric version of the Yang-Mills theory. There is currently much circumstantial evidence for the MO conjecture, but no proof.

It has been suggested by M. Atiyah soon after the work of Goddard, Nuyts and Olive that Langlands duality might be related to the MO duality, but only recently the precise relation has been found [6]. In these lectures I will try to explain the main ideas of [6]. For detailed derivations and a more extensive list of references the reader is referred to the original paper. I will not discuss the ramified version of Geometric Langlands Duality; for that the reader is referred to [7].

2 Electric-magnetic duality in abelian gauge theory

I will begin by reviewing electric-magnetic duality in Maxwell theory, which is a theory of a $U(1)$ gauge field without sources. On the classical level, this theory describes a connection A on a principal $U(1)$ bundle E over a four-manifold X. The four-manifold X is assumed to be equipped with a Lorenzian metric (later we will switch to Riemannian metric). The equations of motion for A read

$$d \star F = 0,$$

where $F = dA$ is the curvature of A and $\star$ is the Hodge star operator on forms. In addition, the curvature 2-form F is closed, $dF = 0$ (this is known as the Bianchi identity), so one can to a large extent eliminate A in favor F. More precisely, F determines the holonomy of A around all contractible loops in X. If $\pi_1(X)$ is trivial, F completely determines A, up to gauge equivalence. In addition, if $H^2(X) \neq 0$, F satisfies a quantization condition: its periods are integral multiples of 2π. The cohomology class of F is the Euler class of E (or alternatively the first Chern class of the associated line bundle).

When $X = \mathbb{R}^{3,1}$, the theory is clearly invariant under a transformation

$$F \mapsto F' = \star F. \tag{1}$$

If X is Lorenzian, this transformation squares to -1. It is known as the electric-magnetic duality. To understand why, let x^0 be the time-like coordinate and $x^i, i = 1, 2, 3$ be space-like coordinates. Then the usual electric and magnetic fields are

$$E_i = F_{0i}, \quad B_i = \frac{1}{2}\epsilon_{ijk}F_{jk},$$

and the transformation (1) acts by

$$E_i \mapsto B_i, \quad B_i \mapsto -E_i.$$

Thus, up to some minus signs, the duality transformation exchanges electric and magnetic fields. The signs are needed for compatibility with Lorenz transformations. Alternatively, from the point of view of the Hamiltonian formalism the signs are needed to preserve the symplectic structure on the space of fields E_i and B_i. This symplectic structure corresponds to the Poisson bracket

$$\{B_i(x), E_k(y)\} = \frac{e^2}{2}\epsilon_{ijk}\partial_j\delta^3(x-y),$$

where e^2 is the coupling constant (it determines the overall normalization of the action).

As remarked above, the classical theory can be rewritten entirely in terms of F only on simply-connected manifolds. In addition, the $\star F$ need not satisfy any quantization condition, unlike F. Thus it appears that on manifolds more complicated than $\mathbb{R}^{3,1}$ the duality is absent. Interestingly, in the quantum theory the duality is restored for any X, if one sums over all topologies of the bundle E. To see how this comes about, let us recall that the quantum theory is defined by its path-integral

$$Z = \int \mathcal{D}A\ e^{iS(A)},$$

where $S(A)$ is the action functional. We take the action to be

$$S(A) = \frac{1}{2e^2}\int_X F\wedge\star F + \frac{\theta}{8\pi^2}\int_X F\wedge F.$$

Its critical points are precisely solutions of $d\star F = 0$. Note that the second term in the action depends only on the topology of E and therefore does not affect the classical equations of motion. But it does affect the action and therefore has to be considered in the quantum theory. In fact, if we sum over all isomorphism classes of E, i.e. define the path-integral as

$$Z = \sum_E \int \mathcal{D}A\ e^{iS(A)},$$

the parameter θ tells us how to weigh contributions of different E.

At this stage it is very convenient to replace X with a Riemannian manifold (which we will also denote X). The idea here is that the path-integral for a Lorenzian manifold should be defined as an analytic continuation of the path-integral in Euclidean signature; this is known as the Wick rotation. In Euclidean signature the path-integral and the action look slightly different:

$$Z = \sum_E \int \mathcal{D}A\ e^{-S_{\mathsf{E}}(A)},$$

where

$$S_{\mathsf{E}}(A) = \int_X \left(\frac{1}{2e^2} F \wedge \star F - \frac{i\theta}{8\pi^2} F \wedge F \right)$$

Note that the action becomes complex in Euclidean signature.

Now let us sketch how duality arises on the quantum level. Assuming that X is simply-connected for simplicity, we can replace integration over A with integration over the space of closed 2-forms F satisfying the quantization condition on periods. If we further assume $X = \mathbb{R}^4$, the quantization condition is empty, and the partition function can be written as

$$Z = \int DFDB \exp \left(-S_{\mathsf{E}} + i \int_X B \wedge dF \right).$$

Here the new field B is a 1-form on X introduced so that integration over it produces the delta-functional $\delta(dF) = \prod_{x \in X} \delta(dF(x))$. This allows us to integrate over all (not necessarily closed) 2-forms F.

Now we can do the integral over F using the fact that it is a Gaussian integral. The result is

$$Z = \int DB \exp \left(-\frac{1}{2\hat{e}^2} \int_X G \wedge \star G + \frac{i\hat{\theta}}{8\pi^2} \int_X G \wedge G \right),$$

where $G = dB$, and the parameters $\hat{e}^2$ and $\hat{\theta}$ are defined by

$$\frac{\hat{\theta}}{2\pi} + \frac{2\pi i}{\hat{e}^2} = -\left(\frac{\theta}{2\pi} + \frac{2\pi i}{e^2} \right)^{-1}.$$

We see that the partition function written as an integral over B has exactly the same form as the partition function written as an integral over A, but with e^2 and θ replaced with $\hat{e}^2$ and $\hat{\theta}$. This is a manifestation of electric-magnetic duality. To see this more clearly, note that for $\theta = 0$ the equations of motion for F deduced from the action

$$S_{\mathsf{E}}(F) - i \int_X B \wedge dF$$

reads

$$F = ie^2 \star G.$$

The factor i arises because of Riemannian signature of the metric; in Lorenzian signature similar manipulations would produce an identical formula but without i.

The above derivation of electric-magnetic duality is valid only when X is topologically trivial. If $H^2(X) \neq 0$, we have to insert additional delta-functions in the path-integral for F and B ensuring that the periods of F are properly quantized. It turns out that the effect of these delta-functions can be reproduced by letting B to be a connection 1-form on an arbitrary principal $U(1)$ bundle $\hat{E}$ over X and summing over all possible $\hat{E}$. For a proof valid for general 4-manifolds X see [8].

We see from this derivation that duality acts nontrivially on the coupling e^2 and the parameter θ. To describe this action, it is convenient to introduce a complex parameter τ taking values in the upper half-plane:

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e^2}.$$

Then electric-magnetic duality acts by

$$\tau \to -1/\tau.$$

Note that the transformation $\tau \to \tau+1$ or equivalently the shift $\theta \to \theta+2\pi$ is also a symmetry of the theory if X is spin. Indeed, θ enters the action as the coefficient of the topological term

$$-\frac{i}{2}c_1^2,$$

where c_1 is the first Chern class of E. If X is spin, the square of any integral cohomology class is divisible by two, and so the above topological term is i times an integer. This immediately implies that shifting θ by 2π leaves e^{-S_E} unchanged. (For arbitrary X the transformation $\tau \to \tau + 2$ is still a symmetry.)

The transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ generated the whole group of integral fractional-linear transformations acting on the upper half-plane, i.e. the group $PSL(2, \mathbb{Z})$. Points in the upper half-plane related by the $PSL(2, \mathbb{Z})$ give rise to isomorphic theories. One may call this group the duality group. Actually, it is better to reserve this name for its "double-cover" $SL(2, \mathbb{Z})$, since applying the electric-magnetic duality twice acts by -1 on the 2-form F. In this lectures we will mostly focus on electric-magnetic duality, which is a particular element of $SL(2,\mathbb{Z})$. It is also known as S-duality. Note that for $\theta = 0$ S-duality exchanges weak coupling ($e^2 \ll 1$) and strong coupling ($e^2 \gg 1$). This does not cause problems in the abelian case, because we can solve the $U(1)$ gauge theory for any value of the coupling. But it will greatly complicate the matters in the nonabelian case, where the theory is only soluble for small e^2.

3 Montonen-Olive Duality

Now let us try to generalize the above considerations to a nonabelian gauge theory, also known as Yang-Mills theory. The basic field is a connection $A = A_\mu dx^\mu$ on a principal G-bundle E over a four-dimensional manifold X. The four-manifold X is equipped with a Lorenzian metric g, and the equations of motion read

$$d_A F = 0, \quad d_A \star F = 0,$$

where $F = dA + A \wedge A \in \Omega^2(\mathrm{ad}(E))$ is the curvature of A, d_A is the covariant differential, and $\star$ is the Hodge star operator on X. The first of these equations is satisfied identically, while the second one follows from varying the Yang-Mills action

$$S(A) = \int_X \left(\frac{1}{2e^2} \mathrm{Tr}\, F \wedge \star F + \frac{\theta}{8\pi^2} \mathrm{Tr}\, F \wedge F \right)$$

The corresponding Euclidean action is

$$S_{\mathsf{E}}(A) = \int_X \left(\frac{1}{2e^2} \mathrm{Tr}\, F \wedge \star F - i\frac{\theta}{8\pi^2} \mathrm{Tr}\, F \wedge F \right)$$

The action has two real parameters, e and θ. Neither of them affects the classical equations of motion, but they do affect the quantized Yang-Mills theory. On the quantum level one should consider a path integral

$$Z = \sum_E \int \mathcal{D}A \; e^{-S_{\mathsf{E}}(A)}, \tag{2}$$

and it does depend on both e and θ.

For each X the path-integral (2) is a single function of e^2 and θ and so is not very informative (it is known as the partition function of Yang-Mills theory). More generally, one can consider path-integrals of the form

$$\sum_E \int \mathcal{D}A \; e^{-S_{\mathsf{E}}(A)} \mathcal{O}_1(A)\mathcal{O}_2(A)\ldots\mathcal{O}_k(A),$$

where $\mathcal{O}_1,\ldots,\mathcal{O}_k$ are gauge-invariant functions of A called observables. Such a path-integral is called a correlator of observables $\mathcal{O}_1,\ldots,\mathcal{O}_k$ and denoted

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_k \rangle.$$

An example of an observable in Yang-Mills theory is

$$\mathcal{O}(A) = W_R(\gamma) = \mathrm{Tr} R(\mathrm{Hol}_\gamma(A)),$$

where γ is a closed curve in X, $\mathrm{Hol}_\gamma(A)$ is the holonomy of A along γ, and R is an irreducible representation of G. Such observables are called Wilson loops [9]; they play an important role in the geometric Langlands duality, as we will see below. From the physical viewpoint, inserting $W_R(\gamma)$ into the path-integral corresponds to inserting an electrically charged source ("quark") in the representation R whose worldline is γ. In semiclassical Yang-Mills theory, such a source creates a Coulomb-like field of the form

$$A_0^a = T^a \, \frac{e^2}{4\pi r}$$

where $T^a, a = 1, \dim G$, are generators of G in representation R. Here we took $X = \mathbb{R} \times \mathbb{R}^3$, r is the distance from the origin in $\mathbb{R}^3$, and we assumed that the worldline of the source is given by $r = 0$.

In the case $G = U(1)$, we saw that the theory enjoys a symmetry which in Lorenzian signature acts by

$$F \to \hat{F} = e^2 \star F, \quad \tau \to \hat{\tau} = -1/\tau.$$

At first sight, it seems unlikely that such a duality could extend to a theory with a nonabelian gauge group, since equations of motion explicitly depend on A, not just on F. The first hint in favor of a nonabelian generalization of electric-magnetic duality was the work of Goddard, Nuyts, and Olive [3]. They noticed that magnetic sources in a nonabelian Yang-Mills theory are labeled by irreducible representations of a different group which they called the magnetic gauge group. As a matter of fact, the magnetic gauge group coincides with the Langlands dual of G, so we will denote it ${}^L G$. A static magnetic source in Yang-Mills theory should create a field of the form

$$F = \star_3 \, d\left(\frac{\mu}{2r}\right),$$

where μ is an element of the Lie algebra $\mathfrak{g}$ of G defined up to adjoint action of G, and $\star_3$ is the Hodge star operator on $\mathbb{R}^3$. Goddard, Nuyts, and Olive showed that μ is "quantized". More precisely, using gauge freedom one can assume that μ lies in a particular Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, and then it turns out that μ must lie in the coweight lattice of G, which, by definition, is the same as the weight lattice of ${}^L G$.[1] Furthermore, μ is defined up to an action of the Weyl group, so possible values of μ are in one-to-one correspondence with highest weights of ${}^L G$.

On the basis of this observation, C. Montonen and D. Olive [4] conjectured that Yang-Mills theories with gauge groups G and ${}^L G$ are isomorphic on the quantum level, and that this isomorphism exchanges electric and magnetic sources. Thus the Montonen-Olive duality is a nonabelian version of electric-magnetic duality in Maxwell theory.

In order for the energy of electric and magnetic sources to transform properly under MO duality, one has to assume that for $\theta = 0$ the dual gauge coupling is

$$\hat{e}^2 = \frac{16\pi^2 n_{\mathfrak{g}}}{e^2}. \tag{3}$$

Here the integer $n_{\mathfrak{g}}$ is 1, 2, or 3 depending on the maximal multiplicity of edges in the Dynkin diagram of $\mathfrak{g}$ [10, 11]; for simply-laced groups $n_{\mathfrak{g}} = 1$. This means that MO duality exchanges weak coupling ($e \to 0$) and strong coupling ($e \to \infty$). For this reason, it is extremely hard to prove the MO duality conjecture. For general θ, we define

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}.$$

(The slight difference in the definition of τ compared to the nonabelian case is due to a different normalization of the Killing metric on the Lie algebra.)

[1]The coweight lattice of G is defined as the lattice of homomorphisms from $U(1)$ to a maximal torus T of G. The weight lattice of G is the lattice of homomorphisms from T to $U(1)$.

The parameter τ takes values in the upper half-plane and under MO duality transforms as

$$\tau \to \hat{\tau} = -\frac{1}{n_{\mathfrak{g}}\tau} \tag{4}$$

The Yang-Mills theory has another, much more elementary symmetry, which does not change the gauge group:

$$\tau \to \tau + k.$$

Here k is an integer which depends on the geometry of X and G. For example, if $X = \mathbb{R}^4$, then $k = 1$ for all G. Together with the MO duality, these transformations generate some subgroup of $SL(2, \mathbb{R})$. In what follows we will mostly set $\theta = 0$ and will discuss only the $\mathbb{Z}_4$ subgroup generated by the MO duality.

To summarize, if the MO duality were correct, then the partition function would satisfy

$$Z(X, G, \tau) = Z(X, {}^LG, -\frac{1}{n_{\mathfrak{g}}\tau})$$

Of course, the partition function is not a very interesting observable. Isomorphism of two quantum field theories means that we should be able to match all observables in the two theories. That is, for any observable O in the gauge theory with gauge group G we should be able to construct an observable $\tilde{O}$ in the gauge theory with gauge group ${}^LG$ so that all correlators agree:

$$\langle O_1 \ldots O_n \rangle_{X,G,\tau} = \langle \tilde{O}_1 \ldots \tilde{O}_n \rangle_{X,{}^LG,-1/(n_{\mathfrak{g}}\tau)}$$

At this point I should come clean and admit that the MO duality as stated above is not correct. The most obvious objection is that the parameters e and $\hat{e}$ are renormalized, and the relation like (3) is not compatible with renormalization. However, it was pointed out later by Osborn [5] (who was building on the work of Witten and Olive [12]) that the duality makes much more sense in $\mathcal{N} = 4$ super-Yang-Mills theory. This is a maximally supersymmetric extension of Yang-Mills theory in four dimensions, and it has a remarkable property that the gauge coupling is not renormalized at any order in perturbation theory. Furthermore, Osborn was able to show that certain magnetically-charged solitons in $\mathcal{N} = 4$ SYM theory have exactly the same quantum numbers as gauge bosons. (The argument assumes that the vacuum breaks spontaneously the gauge group G down to its maximal abelian subgroup, so that both magnetically charged solitons and the corresponding gauge bosons are massive). Later strong evidence in favor of the MO duality for $\mathcal{N}=4$ SYM was discovered by A. Sen [13] and C. Vafa and E. Witten [14]. Nowadays MO duality is often regarded as a consequence of string dualities. One particular derivation which works for all G is explained in [15]. Nevertheless, the MO duality is still a conjecture, not a theorem. In what follows we will assume its validity and deduce from it the main statements of the Geometric Langlands Program.

Apart from the connection 1-form A, $\mathcal{N}=4$ SYM theory contains six scalar fields $\phi_i, i=1,\ldots,6$, which are sections of $\mathrm{ad}(E)$, four spinor fields $\bar{\lambda}_a, a=1,\ldots,4$, which are sections of $\mathrm{ad}(E)\otimes\mathcal{S}_-$ and four spinor fields $\lambda^a, a=1,\ldots,4$ which are sections of $\mathrm{ad}(E)\otimes\mathcal{S}_+$. Here $\mathcal{S}_\pm$ are the two spinor bundles over X. The fields A and ϕ_i are bosonic (even), while the spinor fields are fermionic (odd). In Minkowski signature the fields $\bar{\lambda}_a$ and λ^a are complex-conjugate, but in Euclidean signature they are independent.

The action of $N=4$ SYM theory has the form

$$S_{\mathcal{N}=4} = S_{YM} + \frac{1}{e^2}\int_X \left(\sum_i \mathrm{Tr}\, D\phi_i \wedge \star D\phi_i + \mathrm{vol}_X \sum_{i<j} \mathrm{Tr}[\phi^i,\phi^j]^2\right) + \ldots$$

where dots denote terms depending on the fermions. The action has $Spin(6)\simeq SU(4)$ symmetry under which the scalars ϕ_i transform as a vector, the fields λ_a transform as a spinor, and $\bar{\lambda}^a$ transform as the dual spinor. This symmetry is present for any Riemannian X and is known as the R-symmetry. If X is $\mathbb{R}^4$ with a flat metric, the action also has translational and rotational symmetries, as well as sixteen supersymmetries $\bar{Q}_{a\alpha}$ and $Q^a_{\dot\alpha}$, where $a=1,\ldots,4$ and the $Spin(4)$ spinor indices α and $\dot\alpha$ run from 1 to 2. As is clear from the notation, $\bar{Q}_a$ and Q^a transform as spinors and dual spinors of the R-symmetry group $Spin(6)$; they also transform as spinors and dual spinors of the rotational group $Spin(4)$.

One can show that under the MO duality all bosonic symmetry generators are mapped trivially, while supersymmetry generators are multiplied by a τ-dependent phase:

$$\bar{Q}_a \to e^{i\phi/2}\bar{Q}_a, \quad Q^a \to e^{-i\phi/2}Q^a, \quad e^{i\phi} = \frac{|\tau|}{\tau}$$

This phase will play an important role in the next section.

4 Twisting $\mathcal{N} = 4$ super-Yang-Mills theory

In order to extract mathematical consequences of MO duality, we are going to turn $N = 4$ SYM theory into a topological field theory. The procedure for doing this is called topological twisting [16].

Topological twisting is a two-step procedure. On the first step, one chooses a homomorphism ρ from $Spin(4)$, the universal cover of the structure group of TX, to the R-symmetry group $Spin(6)$. This enables one to redefine how fields transform under $Spin(4)$. The choice of ρ is constrained by the requirement that after this redefinition some of supersymmetries become scalars, i.e. transform trivially under $Spin(4)$. Such supersymmetries survive when X is taken to be an arbitrary Riemannian manifold. In contrast, if we consider ordinary $\mathcal{N} = 4$ SYM on a curved X, it will have supersymmetry only if X admits a covariantly constant spinor.

It is easy to show that there are three inequivalent choices of ρ satisfying these constraints [14]. The one relevant for the Geometric Langlands Program is identifies $Spin(4)$ with the obvious $Spin(4)$ subgroup of $Spin(6)$. After redefining the spins of the fields accordingly, we find that one of the left-handed supersymmetries and one of the right-handed supersymmetries become scalars. We will denote them Q_l and Q_r respectively.

On the second step, one notices that Q_l and Q_r both square to zero and anticommute (up to a gauge transformation). Therefore one may pick any linear combination of Q_l and Q_r

$$Q = uQ_l + vQ_r,$$

and declare it to be a BRST operator. That is, one considers only observables which are annihilated by Q (and are gauge-invariant) modulo those which are Q-exact. This is consistent because any correlator involving Q-closed observables, one of which is Q-exact, vanishes. From now on, all observables are assumed to be Q-closed. Correlators of such observables will be called topological correlators.

Clearly, the theory depends on the complex numbers u, v only up to an overall scaling. Thus we get a family of twisted theories parameterized by the projective line $\mathbb{P}^1$. Instead of the homogenous coordinates u, v, we will mostly use the affine coordinate $t = v/u$ which takes values in $\mathbb{C} \cup \{\infty\}$. All these theories are diffeomorphism-invariant, i.e. do not depend on the Riemannian metric. To see this, one writes an action (which is independent of t) in the form

$$I = \{Q, V\} + \frac{i\Psi}{4\pi} \int_X \mathrm{Tr}\, F \wedge F$$

where V is a gauge-invariant function of the fields, and Ψ is given by

$$\Psi = \frac{\theta}{2\pi} + \frac{t^2 - 1}{t^2 + 1} \frac{4\pi i}{e^2}$$

All the metric dependence is in V, and since changing V changes the action by Q-exact terms, we conclude that topological correlators are independent of the metric.

It is also apparent that for fixed t topological correlators are holomorphic functions of Ψ, and this dependence is the only way e^2 and θ may enter. In particular, for $t = i$ we have $\Psi = \infty$, independently of e^2 and θ. This means that for $t = i$ topological correlators are independent of e^2 and θ.

To proceed further, we need to describe the field content of the twisted theory. Since the gauge field A is invariant under $Spin(6)$ transformations, it is not affected by the twist. As for the scalars, four of them become components of a 1-form ϕ with values in $\mathrm{ad}(E)$, and the other two remain sections of $\mathrm{ad}(E)$; we may combine the latter into a complex scalar field σ which is a section of the complexification of $\mathrm{ad}(E)$. The fermionic fields in the twisted theory are a pair of 1-forms ψ and $\tilde{\psi}$, a pair of 0-forms η and $\tilde{\eta}$, and a 2-form χ, all taking values in the complexification of $\mathrm{ad}(E)$.

What makes the twisted theory manageable is that the path integral localizes on Q-invariant field configurations. One way to deduce this property is to note that as a consequence of metric-independence, semiclassical (WKB) approximation is exact in the twisted theory. Thus the path-integral localizes on absolute minima of the Euclidean action. On the other hand, such configurations are exactly Q-invariant configurations.

The condition of Q-invariance is a set of partial differential equations on the bosonic fields A, ϕ and σ. We will only state the equations for A and ϕ, since the equations for σ generically imply that $\sigma = 0$:

$$(F - \phi \wedge \phi + tD\phi)^+ = 0, \quad (F - \phi \wedge \phi - t^{-1} D\phi)^- = 0, \quad D \star \phi = 0. \tag{5}$$

Here subscripts $+$ and $-$ denote self-dual and anti-self-dual parts of a 2-form.

If t is real, these equations are elliptic. A case which will be of special interest is $t = 1$; in this case the equations can be rewritten as

$$F - \phi \wedge \phi + \star D\phi = 0, \quad D \star \phi = 0.$$

They resemble both the Hitchin equations in 2d [17] and the Bogomolny equations in 3d [18] (and reduce to them in special cases). The virtual dimension of the moduli space of these equations is zero, so the partition function is the only nontrivial observable if X is compact without boundary. However, for applications to the Geometric Langlands Program it is important to consider X which are noncompact and/or have boundaries.

Another interesting case is $t = i$. To understand this case, it is convenient to introduce a complex connection $\mathcal{A} = A + i\phi$ and the corresponding curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A}^2$. Then the equations are equivalent to

$$\mathcal{F} = 0, \quad D \star \phi = 0.$$

The first of these equations is invariant under the complexified gauge transformations, while the second one is not. It turns out that the moduli space is unchanged if one drops the second equation and considers the space of solutions of the equation $\mathcal{F} = 0$ modulo $G_{\mathbb{C}}$ gauge transformations. More precisely, according to a theorem of K. Corlette [19], the quotient by $G_{\mathbb{C}}$ gauge transformations should be understood in the sense of Geometric Invariant Theory, i.e. one should distinguish stable and semistable solutions of $\mathcal{F} = 0$ and impose a certain equivalence relation on semistable solutions. The resulting moduli space is called the moduli space of stable $G_{\mathbb{C}}$ connections on X and will be denoted $\mathcal{M}_{flat}(G, X)$. Thus for $t = i$ the path integral of the twisted theory reduces to an integral over $\mathcal{M}_{flat}(G, X)$. This is an indication that twisted $\mathcal{N} = 4$ SYM with gauge group G has something to do with the study of homomorphisms from $\pi_1(X)$ to $G_{\mathbb{C}}$.

Finally, let us discuss how MO duality acts on the twisted theory. The key observation is that MO duality multiplies Q_l and Q_r by $e^{\pm i\phi/2}$, and therefore multiplies t by a phase:

$$t \mapsto \frac{|\tau|}{\tau} t.$$

Since $\mathrm{Im}\tau \neq 0$, the only points of the $\mathbb{P}^1$ invariant under the MO duality are the "poles" $t = 0$ and $t = \infty$. On the other hand, if we take $t = i$ and $\theta = 0$, then the MO duality maps it to a theory with $t = 1$ and $\theta = 0$ (it also replaces G with ${}^LG$). As we will explain below, it is this special case of the MO duality that gives rise to the Geometric Langlands Duality.

5 Reduction to two dimensions

From now on we specialize to the case $X = \Sigma \times C$ where C and Σ are Riemann surfaces. We will assume that C has no boundary and has genus $g > 1$, while Σ may have a boundary. In our discussion we will mostly work locally on Σ, and its global structure will be unimportant.

Topological correlators are independent of the volumes of C and Σ. However, to exploit localization, it is convenient to consider the limit in which the volume of C goes to zero. In the spirit of the Kaluza-Klein reduction, we expect that in this limit the 4d theory becomes equivalent to a 2d theory on Σ. In the untwisted theory, this equivalence holds only in the limit $\mathrm{vol}(C) \to 0$, but in the twisted theory the equivalence holds for any volume.

It is easy to guess the effective field theory on Σ. One begins by considering the case $\Sigma = \mathbb{R}^2$ and requiring the field configuration to be independent of the coordinates on Σ and to have zero energy. On can show that a generic such configuration has $\sigma = 0$, while ϕ and A are pulled-back from C and satisfy

$$F - \phi \wedge \phi = 0, \quad D\phi = 0, \quad D \star \phi = 0.$$

Here all quantities as well as the Hodge star operator refer to objects living on C. These equations are known as Hitchin equations [17], and their space of solutions modulo gauge transformations is called the Hitchin moduli space $\mathcal{M}_H(G, C)$. The space $\mathcal{M}_H(G, C)$ is a noncompact manifold of dimension $4(g-1)\dim G$ with singularities.[2] From the physical viewpoint, $\mathcal{M}_H(G, C)$ is the space of classical vacua of the twisted $\mathcal{N} = 4$ SYM on $C \times \mathbb{R}^2$.

In the twisted theory, only configurations with vanishingly small energies contribute. In the limit $\mathrm{vol}(C) \to 0$, such configurations will be represented by slowly varying maps from Σ to $\mathcal{M}_H(G, C)$. Therefore we expect the effective field theory on Σ to be a topological sigma-model with target $\mathcal{M}_H(G, C)$.

Before we proceed to identify more precisely this topological sigma-model, let us note that $\mathcal{M}_H(G, C)$ has singularities coming from solutions of Hitchin equations which are invariant under a subgroup of gauge transformations. In the neighborhood of such a classical vacuum, the effective field theory is not equivalent to a sigma-model, because of unbroken gauge symmetry. In fact, it is difficult to describe the physics around such vacua in purely 2d terms. We will avoid this difficulty by imposing suitable conditions on the boundary of Σ ensuring that we stay away from such dangerous points.

[2] We assumed $g > 1$ precisely to ensure that virtual dimension of $\mathcal{M}_H(G, C)$ is positive.

The most familiar examples of topological sigma-models are A and B-models associated to a Calabi-Yau manifold M [20]. Both models are obtained by twisting a supersymmetric sigma-model with target M. The path-integral of the A-model localizes on holomorphic maps from Σ to M and computes the Gromov-Witten invariants of M. The path-integral for the B-model localizes on constant maps to M and can be interpreted mathematically in terms of deformation theory of M regarded as a complex manifold [21]. Both models are topological field theories (TFTs), in the sense that correlators do not depend on the metric on Σ. In addition, the A-model depends on the symplectic structure of M, but not on its complex structure, while the B-model depends on the complex structure of M, but not on its symplectic structure.

As explained above, we expect that our family of 4d TFTs, when considered on a 4-manifold of the form $\Sigma \times C$, becomes equivalent to a family of topological sigma-models with target $\mathcal{M}_H(G, C)$. To connect this family to ordinary A and B-models, we note that $\mathcal{M}_H(G, C)$ is a (noncompact) hyper-Kähler manifold. That is, it has a $\mathbb{P}^1$ worth of complex structures compatible with a certain metric. This metric has the form

$$ds^2 = \frac{1}{e^2} \int_C \mathrm{Tr}\, (\delta A \wedge \star \delta A + \delta \phi \wedge \star \delta \phi)$$

where $(\delta A, \delta \phi)$ is a solution of the linearized Hitchin equations representing a tangent vector to $\mathcal{M}_H(G, C)$.[3] If we parameterize the sphere of complex structures by a parameter $w \in \mathbb{C} \cup \{\infty\}$, the basis of holomorphic differentials is

$$\delta A_{\bar{z}} - w\, \delta \phi_{\bar{z}}, \quad \delta A_z + w^{-1} \delta \phi_z.$$

By varying w, we get a family of B-models with target $\mathcal{M}_H(G, C)$. Similarly, since for each w we have the corresponding Kähler form on $\mathcal{M}_H(G, C)$, by varying w we get a family of A-models with target $\mathcal{M}_H(G, C)$. However, the family of topological sigma-models obtained from the twisted $\mathcal{N} = 4$ SYM does not coincide with either of these families. The reason for this is that a generic A-model or B-model with target $\mathcal{M}_H(G, C)$ depends on the complex structure on C, and therefore cannot arise from a TFT on $\Sigma \times C$.

As explained in [6], this puzzle is resolved by recalling that for a hyper-Kähler manifold M there are twists other than ordinary A or B twists. In

[3] The overall normalization of the metric we use is natural from the point of view of gauge theory.

general, twisting a supersymmetric sigma-model requires picking two complex structures on the target. If we are given a Kähler structure on M, one can choose the two complex structures to be the same (B-twist) or opposite (A-twist). But for a hyper-Kähler manifold there is a whole sphere of complex structures, and by independently varying the two complex structures one gets $\mathbb{P}^1 \times \mathbb{P}^1$ worth of 2d TFTs. They are known as generalized topological sigma-models, since their correlators depend on a generalized complex structure on the target M [22, 23]. The notion of a generalized complex structure was introduced by N. Hitchin [24] and it includes complex and symplectic structures are special cases.

It turns out that for $M = \mathcal{M}_H(G,C)$ there is a 1-parameter subfamily of this 2-parameter family of topological sigma-models which does not depend on the complex structure or Kähler form of C. It is this subfamily which appears as a reduction of the twisted $\mathcal{N} = 4$ SYM theory. Specifically, the two complex structures on $\mathcal{M}_H(G,C)$ are given by

$$w_+ = -t, \quad w_- = t^{-1}$$

Note that one gets a B-model if and only if $w_+ = w_-$, i.e. if $t = \pm i$. One gets an A-model if and only if t is real. All other values of t correspond to generalized topological sigma-models.

Luckily to understand Geometric Langlands Duality we mainly need the two special cases $t = i$ and $t = 1$. The value $t = i$ corresponds to a B-model with complex structure J defined by complex coordinates

$$A_z + i\phi_z, \quad A_{\bar{z}} + i\phi_{\bar{z}}$$

on $\mathcal{M}_H(G,C)$. These are simply components of the complex connection $\mathcal{A} = A + i\phi$ along C. In terms of this complex connection two out of three Hitchin equations are equivalent to

$$\mathcal{F} = d\mathcal{A} + \mathcal{A}^2 = 0$$

This equation is invariant under complexified gauge transformations. The third equation $D \star \phi = 0$ is invariant only under G gauge transformations. By a theorem of S. Donaldson [25], one can drop this equation at the expense of enlarging the gauge group from G to $G_{\mathbb{C}}$.(More precisely, one also has to identify certain semistable solutions of the equation $\mathcal{F} = 0$.) This is analogous to the situation in four dimensions. Thus in complex structure J the moduli space $\mathcal{M}_H(G,C)$ can be identified with the moduli space $\mathcal{M}_{flat}(G,C)$ of stable flat $G_{\mathbb{C}}$ connections on C. It is apparent that J is independent of the complex structure on C, which implies that the B-model at $t=i$ is also independent of it.

The value $t=1$ corresponds to an A-model with a symplectic structure $\omega = 4\pi\omega_K/e^2$, where

$$\omega_K = -\frac{1}{2\pi}\int_C \mathrm{Tr}\,\delta\phi \wedge \delta A.$$

It is a Kähler form of a certain complex structure K on $\mathcal{M}_H(G,C)$. Note that ω_K is exact and independent of the complex structure on C.

Yet another complex structure on $\mathcal{M}_H(G,C)$ is $I=JK$. It will make an appearance later, when we discuss Homological Mirror Symmetry for $\mathcal{M}_H(G,C)$. In this complex structure, $\mathcal{M}_H(G,C)$ can be identified with the moduli space of *stable holomorphic Higgs bundles*. Recall that a (holomorphic) Higgs bundle over C (with gauge group G) is a holomorphic G-bundle E over C together with a holomorphic section φ of $\mathrm{ad}(E)$. In complex dimension one, any principal G bundle can be thought of as a holomorphic bundle, and Hitchin equations imply that $\varphi = \phi^{1,0}$ satisfies

$$\bar{\partial}\varphi = 0.$$

This gives a map from $\mathcal{M}_H(G,C)$ to the set of gauge-equivalence classes of Higgs bundles. This map becomes one-to-one if we limit ourselves to stable or semistable Higgs bundles and impose a suitable equivalence relationship on semistable ones. This gives an isomorphism between $\mathcal{M}_H(G,C)$ and $\mathcal{M}_{Higgs}(G,C)$.

It is evident that the complex structure I, unlike J, does depend on the choice of complex structure on C. Therefore the B-model for the moduli space of stable Higgs bundles cannot be obtained as a reduction of a 4d TFT.[4] On the other hand, the A-model for $\mathcal{M}_{Higgs}(G,C)$ is independent of the choice of complex structure on C, because the Kähler form ω_I is given by

$$\omega_I = \frac{1}{4\pi}\int_C \mathrm{Tr}\,(\delta A \wedge \delta A - \delta\phi \wedge \delta\phi).$$

In fact, the A-model for $\mathcal{M}_{Higgs}(G,C)$ is obtained by letting $t=0$. This special case of reduction to 2d has been first discussed in [27].

[4] It can be obtained as a reduction of a "holomorphic-topological" gauge theory on $\Sigma \times C$ [26].

6 Mirror Symmetry for the Hitchin moduli space

Now we are ready to infer the consequences of the MO duality for the topological sigma-model with target $\mathcal{M}_H(G,C)$. For $\theta = 0$, the MO duality identifies twisted $\mathcal{N} = 4$ SYM theory with gauge group ${}^LG$ and $t = i$ with a similar theory with gauge group G and $t = 1$. Therefore the B-model with target $\mathcal{M}_{flat}({}^LG, C)$ and the A-model with target $(\mathcal{M}(G,C), \omega_K)$ are isomorphic.

Whenever we have two Calabi-Yau manifolds M and M' such that the A-model of M is equivalent to the B-model of M', we say that M and M' are a mirror pair. Thus MO duality implies that $\mathcal{M}_{flat}({}^LG, C)$ and $\mathcal{M}_H(G,C)$ (with the symplectic structure ω_K) are a mirror pair. This mirror symmetry was first proposed in [28].

The most obvious mathematical interpretation of this statement involves the isomorphism of two Frobenius manifolds associated to the A-model of $(\mathcal{M}_H(G,C), \omega_K)$ and the B-model of $\mathcal{M}_{flat}({}^LG, C)$. The former of these encodes the Gromov-Witten invariants of $\mathcal{M}_H(G,C)$, while the latter one has to to with the complex structure deformations of $\mathcal{M}_{flat}({}^LG, C)$. But one can get a much stronger statement by considering the categories of topological D-branes associated to the two models.

Recall that a topological D-brane for a 2d TFT is a BRST-invariant boundary condition for it. The set of all topological D-branes has a natural structure of a category (actually, an A_∞ category). For a B-model with a Calabi-Yau target M, this category is believed to be equivalent to the derived category of coherent sheaves on M. Sometimes we will also refer to it as the category of B-branes on M. For an A-model with target M', we get a category of A-branes on M'. It contains the derived Fukaya category of M' as a full subcategory. For a review of these matters, see e.g. [29].

It has been argued by A. Strominger, S-T. Yau and E. Zaslow (SYZ) [30] that whenever M and M' are mirror to each other, they should admit Lagrangian torus fibrations over the same base $\mathbf{B}$, and these fibrations are dual to each other, in a suitable sense. In the case of Hitchin moduli spaces, the SYZ fibration is easy to identify [6]. One simply maps a solution (A, ϕ) of the Hitchin equations to the space of gauge-invariant polynomials built from $\varphi = \phi^{1,0}$. For example, for $G = SU(N)$ or $G = SU(N)/\mathbb{Z}_N$ the algebra of gauge-invariant polynomials is generated by

$$\mathcal{P}_n = \operatorname{Tr} \varphi^n \in H^0(C, K_C^n), \quad n = 2, \ldots, N,$$

so the fibration map maps $\mathcal{M}_H(G,C)$ to the vector space

$$\oplus_{n=2}^N H^0(C, K_C^n).$$

The map is surjective [31], so this vector space is the base space $\mathbf{B}$. The map to $\mathbf{B}$ is known as the Hitchin fibration of $\mathcal{M}_H(G,C)$. It is holomorphic in complex structure I and its fibers are Lagrangian in complex structures J and K. In fact, one can regard $\mathcal{M}_{Higgs}(G,C)$ as a complex integrable system, in the sense that the generic fiber of the fibration is a complex torus which is Lagrangian with respect to a holomorphic symplectic form on $\mathcal{M}_{Higgs}(G,C)$

$$\Omega_I = \omega_J + i\omega_K.$$

(Ordinarily, an integrable system is associated with a real symplectic manifold with a Lagrangian torus fibration.)

For general G one can define the Hitchin fibration in a similar way, and one always finds that the generic fiber is a complex Lagrangian torus. Moreover, the bases $\mathbf{B}$ and ${}^L\mathbf{B}$ of the Hitchin fibrations for $\mathcal{M}_H(G,C)$ and $\mathcal{M}_H({}^LG,C)$ are naturally identified.[5]

While the Hitchin fibration is an obvious candidate for the SYZ fibration, can we prove that it really *is* the SYZ fibration? It turns out this statement can be deduced from some additional facts about MO duality.

While we do not know how the MO duality acts on general observables in the twisted theory, the observables $\mathcal{P}_n$ are an exception, as their expectation values parameterize the moduli space of vacua of the twisted theory on $X = \mathbb{R}^4$. The MO duality must identify the moduli spaces of vacua, in a way consistent with other symmetries of the theory, and this leads to a unique identification of the algebras generated by $\mathcal{P}_n$ for G and ${}^LG$. See [6] for details.

To complete the argument, we need to consider some particular topological D-branes for $\mathcal{M}_H(G,C)$ and $\mathcal{M}_H({}^LG,C)$. The simplest example of a B-brane on $\mathcal{M}_{flat}({}^LG,C)$ is the structure sheaf of a (smooth) point. What is its mirror? Since each point p lies in some fiber ${}^L\mathbf{F}_p$ of the Hitchin fibration,

[5] In some cases G and ${}^LG$ coincide, but the relevant identification is not necessarily the identity map [11, 6].

its mirror must be an A-brane on $\mathcal{M}_H(G, C)$ localized on the corresponding fiber $\mathbf{F}_p$ of the Langlands-dual fibration. By definition, this means that the Hitchin fibration is the same as the SYZ fibration.

According to Strominger, Yau, and Zaslow, the fibers of the two mirror fibrations over the same base point are T-dual to each other. Indeed, by the usual SYZ argument the A-brane on $\mathbf{F}_p$ must be a rank-one object of the Fukaya category, i.e. a flat unitary rank-1 connection on a topologically trivial line bundle over $\mathbf{F}_p$. The moduli space of such objects must coincide with the moduli space of the mirror B-brane, which is simply ${}^L\mathbf{F}_p$. This is precisely what we mean by saying that that ${}^L\mathbf{F}_p$ and $\mathbf{F}_p$ are T-dual.

In the above discussion we have tacitly assumed that both $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$ are connected. The components of $\mathcal{M}_H(G, C)$ are labeled by the topological isomorphism classes of principal G-bundles over C, i.e. by elements of $H^2(C, \pi_1(G)) = \pi_1(G)$. Thus, strictly speaking, our discussion applies literally only when both G and ${}^LG$ are simply-connected. This is rarely true; for example, among compact simple Lie groups only E_8, F_4 and G_2 satisfy this condition (all these groups are self-dual).

In general, to maintain mirror symmetry between $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$, one has to consider all possible flat B-fields on both manifolds. A flat B-field on M is a class in $H^2(M, U(1))$ whose image in $H^3(M, \mathbb{Z})$ under the Bockstein homomorphism is trivial. In the case of $\mathcal{M}_H(G, C)$, the allowed flat B-fields have finite order; in fact, they take values in $H^2(C, Z(G)) = Z(G)$, where $Z(G)$ is the center of G. It is well known that $Z(G)$ is naturally isomorphic to $\pi_1({}^LG)$. One can show that MO duality maps the class $w \in Z(G)$ defining the B-field on $\mathcal{M}_H(G, C)$ to the corresponding element in $\pi_1({}^LG)$ labeling the connected component of $\mathcal{M}_H({}^LG, C)$ (and vice versa). For example, if $G = SU(N)$, then $\mathcal{M}_H(G, C)$ is connected and has N possible flat B-fields labeled by $Z(SU(N)) = \mathbb{Z}_N$. On the other hand, ${}^LG = SU(N)/\mathbb{Z}_N$, and therefore $\mathcal{M}_H({}^LG, C)$ has N connected components labeled by $\pi_1({}^LG) = \mathbb{Z}_N$. There can be no nontrivial flat B-field on $\mathcal{M}_H({}^LG, C)$ in this case. See section 7 of [6] for more details.

Let us summarize what we have learned so far. MO duality implies that $\mathcal{M}_H(G, C)$ and $\mathcal{M}_H({}^LG, C)$ are a mirror pair, with the SYZ fibrations being the Hitchin fibrations. The most powerful way to formulate the statement of mirror symmetry between two Calabi-Yau manifolds is in terms of the corresponding categories of topological branes. In the present case, we get that the derived category of coherent sheaves on $\mathcal{M}_{flat}({}^LG, C)$ is equivalent to the category of A-branes on $\mathcal{M}_H(G, C)$ (with respect to the exact symplectic form ω_K). Furthermore, this equivalence maps a (smooth) point p belonging to the fiber ${}^L\mathbf{F}_p$ of the Hitchin fibration of $\mathcal{M}_{flat}({}^LG, C)$ to the Lagrangian submanifold of $\mathcal{M}_H(G, C)$ given by the corresponding fiber of the dual fibration of $\mathcal{M}_H(G, C)$. The flat unitary connection on F_p is determined by the position of p on ${}^L\mathbf{F}_p$.[6]

7 From A-branes to D-modules

Geometric Langlands Duality says that the derived category of coherent sheaves on $\mathcal{M}_{flat}({}^LG, C)$ is equivalent to the derived category of D-modules on the moduli stack $\mathrm{Bun}_G(C)$ of holomorphic G-bundles on C. This equivalence is supposed to map a point on $\mathcal{M}_{flat}({}^LG, C)$ to a Hecke eigensheaf on $\mathrm{Bun}_G(C)$. We have seen that MO duality implies a similar statement, with A-branes on $\mathcal{M}_H(G, C)$ taking place of objects of the derived category of D-modules on $\mathrm{Bun}_G(C)$. Our first goal is to explain the connection between A-branes on $\mathcal{M}_H(G, C)$ and D-modules on $\mathrm{Bun}_G(C)$. Later we will see how the Hecke eigensheaf condition can be interpreted in terms of A-branes.

The starting point of our argument is a certain special A-brane on $\mathcal{M}_H(G, C)$ which was called the canonical coisotropic brane in [6]. Recall that a submanifold Y of a symplectic manifold M is called coisotropic if for any $p \in Y$ the skew-complement of TY_p in TM_p is contained in TY_p. A coisotropic submanifold of M has dimension larger or equal than half the dimension of M; a Lagrangian submanifold of M can be defined as a middle-dimensional coisotropic submanifold. While the most familiar examples of A-branes are Lagrangian submanifolds equipped with flat unitary vector bundles, it is known from the work [32] that the category of A-branes may contain more general coisotropic submanifolds with non-flat vector bundles. (Because of this, in general the Fukaya category is only a full subcategory of the category of A-branes.) The conditions on the curvature of a vector bundle on a coisotropic A-brane are not understood except in the rank-one case; even in this case they are fairly complicated [32]. Luckily, for our purposes we only need the special case when $Y = M$ and the vector bundle has rank one. Then the condition on the curvature 2-form $F \in \Omega^2(M)$ is

$$(\omega^{-1}F)^2 = -1. \tag{6}$$

[6]This makes sense only if ${}^L\mathbf{F}_p$ is smooth. If p is a smooth point but ${}^L\mathbf{F}_p$ is singular, it is not clear how to identify the mirror A-brane on $\mathcal{M}_H(G, C)$.

Here we regard both F and the symplectic form ω as bundle morphisms from TM to T^*M, so that $I_F = \omega^{-1}F$ is an endomorphism of TM.

The condition (6) says that I_F is an almost complex structure. Using the fact that ω and F are closed 2-forms, one can show that I_F is automatically integrable [32].

Let us now specialize to the case $M = \mathcal{M}_H(G, C)$ with the symplectic form $\omega = 4\pi\omega_K/e^2$. Then if we let

$$F = \frac{4\pi}{e^2}\omega_J = \frac{2}{e^2}\int_C \mathrm{Tr}\,\delta\phi \wedge \star\delta A,$$

the equation is solved, and

$$I_F = \omega_K^{-1}\omega_J = I.$$

Furthermore, since F is exact:

$$F \sim \delta \int_C \mathrm{Tr}\,\phi \wedge \star\delta A,$$

we can regard F as the curvature of a unitary connection on a trivial line bundle. This connection is defined uniquely if $\mathcal{M}_H(G, C)$ is simply connected; otherwise any two such connections differ by a flat connection. One can show that flat $U(1)$ connections on a connected component of $\mathcal{M}_H(G, C)$ are classified by elements of $H^1(C, \pi_1(G))$ [6]. Thus we obtain an almost canonical coisotropic A-brane on $\mathcal{M}_H(G, C)$: it is unique up to a finite ambiguity, and its curvature is completely canonical.

Next we need to understand the algebra of endomorphisms of the canonical coisotropic brane. From the physical viewpoint, this is the algebra of vertex operators inserted on the boundary of the worldsheet Σ; such vertex operators are usually referred to as open string vertex operators (as opposed to closed string vertex operators which are inserted at interior points of Σ).

In the classical limit, BRST-invariant vertex operators of ghost number zero are simply functions on the target X holomorphic in the complex structure $I_F = \omega^{-1}F$ [32]. In the case of the canonical coisotropic A-brane, the target $X = \mathcal{M}_H(G, C)$ in complex structure $I_F = I$ is isomorphic to the moduli space of Higgs bundles on C. BRST-invariant vertex operators of higher ghost number can be identified with the Dolbeault cohomology of the moduli space of Higgs bundles.

Actually, the knowledge of the algebra turns out to be insufficient: we would like to work locally in the target space $\mathcal{M}_H(G, C)$ and work with a sheaf of open string vertex operators on $\mathcal{M}_H(G, C)$. The idea of localizing in target space has been previously used by F. Malikov, V. Schechtman and A. Vaintrob to define the chiral de Rham complex [33]; we need an open-string version of this construction.

Localizing the path-integral in target space makes sense only if nonperturbative effects can be neglected [34, 35]. The reason is that perturbation theory amounts to expanding about constant maps from Σ to M, and therefore the perturbative correlator is an integral over M of a quantity whose value at a point $p \in M$ depends only of the infinitesimal neighborhood of p. In such a situation it makes sense to consider open-string vertex operators defined only locally on M, thereby getting a sheaf on M. Because of the topological character of the theory, the OPE of Q-closed vertex operators is nonsingular, and Q-cohomology classes of such locally-defined vertex operators form a sheaf of algebras on M. One difference compared to the closed-string case is that operators on the boundary of Σ have a well-defined cyclic order, and therefore the multiplication of vertex operators need not be commutative. The cohomology of this sheaf of algebras is the endomorphism algebra of the brane [6].

One can show that there are no nonperturbative contributions to any correlators involving the canonical coisotropic A-brane [6], and so one can localize the path-integral in $\mathcal{M}_H(G, C)$. But a further problem arises: perturbative results are formal power series in the Planck constant, and there is no guarantee of convergence. In the present case, the role of the Planck constant is played by the parameter e^2 in the gauge theory.[7] In fact, one can show that the series defining the multiplication have zero radius of convergence for some locally defined observables.

In order to understand the resolution of this problem, let us look more closely the structure of the perturbative answer. In the classical approximation (leading order in e^2), the sheaf of open-string states is quasiisomorphic, as a sheaf of algebras, to the sheaf of holomorphic functions on $\mathcal{M}_{Higgs}(G, C)$. The natural holomorphic coordinates on $\mathcal{M}_{Higgs}(G, C)$ are $A_{\bar{z}}$ and ϕ_z. The algebra of holomorphic functions has an obvious grading in which ϕ_z has

[7]At first sight, the appearance of e^2 in the twisted theory may seem surprising, but one should remember that the argument showing that the theory at $t = 1$ is independent of e^2 is valid only when the manifold X has no boundary.

degree 1 and $A_{\bar{z}}$ has degree 0. Note also that the projection $(A_{\bar{z}}, \phi_z) \mapsto A_{\bar{z}}$ defines a map from $\mathcal{M}_H(G,C)$ to $\mathrm{Bun}_G(C)$. If we restrict the target of this map to the subspace of stable G-bundles $\mathcal{M}(G,C)$, then the preimage of $\mathcal{M}(G,C)$ in $\mathcal{M}_H(G,C)$ can be thought of as the cotangent bundle to $\mathcal{M}(G,C)$.

At higher orders, the multiplication of vertex operators becomes noncommutative and incompatible with the grading. However, it is still compatible with the associated filtration. That is, the product of two functions on $\mathcal{M}_{Higgs}(G,C)$ which are polynomials in ϕ_z of degrees k and l is a polynomial of degree $k+l$. Therefore the product of polynomial observables defined by perturbation theory is well-defined (it is a polynomial in the Planck constant).

We see that we can get a well-defined multiplication of vertex operators if we restrict to those which depend polynomially on ϕ_z. That is, we have to "sheafify" our vertex operators only along the base of the projection to $\mathcal{M}(G,C)$, while along the fibers the dependence is polynomial.

Holomorphic functions on the cotangent bundle of $\mathcal{M}(G,C)$ polynomially depending on the fiber coordinates can be thought of as symbols of differential operators acting on holomorphic functions on $\mathcal{M}(G,C)$, or perhaps on holomorphic sections of a line bundle on $\mathcal{M}(G,C)$. One may therefore suspect that the sheaf of open-string vertex operators is isomorphic to the sheaf of holomorphic differential operators on some line bundle $\mathcal{L}$ over $\mathcal{M}(G,C)$. To see how this comes about, we note that the action of the A-model on a Riemann surface Σ can be written as

$$S = \int_\Sigma \Phi^*(\omega - iF) + \mathrm{BRST} - \text{exact terms}.$$

Here Φ is a map from Σ to $\mathcal{M}(G,C)$ (the basic field of the sigma-model). In our case, both ω and F are exact and the integral reduces to the integral over the boundary of Σ. More precisely, let Ω_I denote the holomorphic symplectic form

$$\Omega_I = -\frac{1}{\pi}\int_C \mathrm{Tr}\left(\delta\phi_z \delta A_{\bar{z}}\right)$$

on $\mathcal{M}_{Higgs}(G,C)$. This form is exact: $\Omega_I = d\varpi$, where

$$\varpi = -\frac{1}{\pi}\int_C \mathrm{Tr}\left(\phi_z \delta A_{\bar{z}}\right).$$

Then the action has the form

$$S = \operatorname{Im}\tau \int_{\partial\Sigma} \Phi^*\varpi + \mathrm{BRST} - \text{exact terms}.$$

Next we note that under the birational identification of $\mathcal{M}_{Higgs}(G,C)$ with the cotangent bundle of $\mathcal{M}(G,C)$, the form ϖ becomes the canonical holomorphic 1-form pdq on the cotangent bundle. Thus the path-integral for the A-model is very similar to the path-integral of a particle on $\mathcal{M}(G,C)$ with the zero Hamiltonian, with $-i\mathrm{Im}\tau$ playing the role of the inverse Planck constant. The main difference is that instead of arbitrary functions on the cotangent bundle in the A-model one only considers holomorphic functions. Otherwise, quantization proceeds in much the same way, and one finds that the algebra of vertex operators can be quantized into the algebra of holomorphic differential operators on $\mathcal{M}(G,C)$. Here the usual quantization ambiguity creeps in: commutation relations

$$[p_i, q^j] = \operatorname{Im}\tau\, \delta_i^j, \quad [p_i, p_j] = [q_i, q_j] = 0$$

can be represented by holomorphic differential operators on an arbitrary complex power of a holomorphic line bundle on $\mathcal{M}(G,C)$. To fix this ambiguity, one can appeal to the additional discrete symmetry of the problem: the symmetry under "time-reversal". This symmetry reverses orientation of Σ and also multiplies ϕ_z by -1. If we want to maintain this symmetry on the quantum level, we must require that the algebra of vertex operators be isomorphic to its opposite algebra. It is known that this is true precisely for holomorphic differential operators on the square root of the canonical line bundle of $\mathcal{M}(G,C)$ [37]. We conclude that the quantized algebra of vertex operators is isomorphic to the sheaf of holomorphic differential operators on $K^{1/2}$, where K is the canonical line bundle on $\mathcal{M}(G,C)$.

Now we can finally explain the relation between A-branes and (twisted) D-modules on $\mathcal{M}(G,C) \subset \mathrm{Bun}_G(C)$. Given an A-brane β, we can consider the space of morphisms from the canonical coisotropic brane α to the brane β. It is a left module over the endomorphism algebra of α. Better still, we can sheafify the space of morphisms along $\mathcal{M}(G,C)$ and get a sheaf of modules over the sheaf of differential operators on $K^{1/2}$, where K is the canonical line bundle of $\mathcal{M}(G,C)$. This is the twisted D-module corresponding to the brane β.

In general it is rather hard to determine the D-module corresponding to a particular A-brane. A simple case is when β is a Lagrangian submanifold defined by the condition $\phi = 0$, i.e. the zero section of the cotangent bundle $\mathcal{M}(G, C)$. In that case, the D-module is simply the sheaf of sections of $K^{1/2}$. From this example, one could suspect that the A-brane is simply the characteristic variety of the corresponding D-module. However, this is not so in general, since in general A-branes are neither conic nor even holomorphic subvarieties of $\mathcal{M}_{Higgs}(G, C)$. For example, a fiber of the Hitchin fibration $\mathbf{F}_p$ is holomorphic but not conic. It is not clear how to compute the D-module corresponding to $\mathbf{F}_p$, even when $\mathbf{F}_p$ is a smooth fiber.[8]

We conclude this sections with two remarks. First, the relation between A-branes is most readily understood if we replace the stack $\mathrm{Bun}_G(C)$ by the space of stable G-bundles $\mathcal{M}(G, C)$. Second, from the physical viewpoint it is more natural to work directly with A-branes rather than with corresponding D-modules. In some sense it is also more natural from the mathematical viewpoint, since both the derived category of $\mathcal{M}_{flat}({}^L G, C)$ and the category of A-branes on $\mathcal{M}_H(G, C)$ are "topological", in the sense that they do not depend on the complex structure on C. The complex structure on C appears only when we introduce the canonical coisotropic brane (its curvature F manifestly depends on the Hodge star operator on C).

8 Wilson and 't Hooft operators

In any gauge theory one can define Wilson loop operators:

$$\mathrm{Tr}_R\, P \exp \int_\gamma A = \mathrm{Tr}\, R\left(\mathrm{Hol}(A, \gamma)\right),$$

where R is a finite-dimensional representation of G, γ is a closed loop in M, and $P \exp \int$ is simply a physical notation for holonomy. The Wilson loop is a gauge-invariant function of the connection A and therefore can be regarded as a physical observable. Inserting the Wilson loop into the path-integral is equivalent to inserting an infinitely massive particle traveling along the path γ and having internal "color-electric" degrees of freedom described by representation R of G. For example, in the theory of strong nuclear interactions we have $G = SU(3)$, and the effect of a massive quark can be modeled by a Wilson loop with R being a three-dimensional irreducible representation. The vacuum expectation value of the Wilson loop can be

[8] The abelian case $G = U(1)$ is an exception, see [6] for details.

used to distinguish various massive phases of the gauge theory [9]. Here however we will be interested in the algebra of Wilson loop operators, which is insensitive to the long-distance properties of the theory.

The Wilson loop is not BRST-invariant and therefore is not a valid observable in the twisted theory. But it turns out that for $t = \pm i$ there is a simple modification which *is* BRST-invariant:

$$W_R(\gamma) = \mathrm{Tr}_R\, P \exp \int_\gamma (A \pm i\phi) = \mathrm{Tr}\, R\left(\mathrm{Hol}(A \pm i\phi), \gamma)\right)$$

The reason is that the complex connection $\mathcal{A} = A \pm i\phi$ is BRST-invariant for these values of t. There is nothing similar for any other value of t.

We may ask how the MO duality acts on Wilson loop operators. The answer is to a large extent fixed by symmetries, but turns out to be rather nontrivial [38]. The difficulty is that the dual operator cannot be written as a function of fields, but instead is a *disorder operator.* Inserting a disorder operator into the path-integral means changing the space of fields over which one integrates. For example, a disorder operator localized on a closed curve γ is defined by specifying a singular behavior for the fields near γ. The disorder operator dual to a Wilson loop has been first discussed by G. 't Hooft [39] and is defined as follows [38]. Let μ be an element of the Lie algebra $\mathfrak{g}$ defined up to adjoint action of G, and let us choose coordinates in the neighborhood of a point $p \in \gamma$ so that γ is defined by the equations $x^1 = x^2 = x^3 = 0$. Then we require the curvature of the gauge field to have a singularity of the form

$$F \sim \star_3\, d\left(\frac{\mu}{2r}\right),$$

where r is the distance to the origin in the 123 hyperplane, and $\star_3$ is the Hodge star operator in the same hyperplane. For $t = 1$ Q-invariance requires the 1-form Higgs field ϕ to be singular as well:

$$\phi \sim \frac{\mu}{2r}\, dx^4.$$

One can show that such an ansatz for F makes sense (i.e. one can find a gauge field whose curvature is F) if and only if μ is a Lie algebra homomorphism from $\mathbb{R}$ to $\mathfrak{g}$ obtained from a Lie group homomorphism $U(1) \to G$ [3]. To describe this condition in a more suggestive way, let us use the gauge freedom to conjugate μ to a particular Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then μ must lie in the coweight lattice $\Lambda_{cw}(G) \subset \mathfrak{t}$, i.e. the lattice of homomorphisms from $U(1)$

to the maximal torus T corresponding to $\mathfrak{t}$. In addition, one has to identify points of the lattice which are related by an element of the Weyl group $\mathcal{W}$. Thus 't Hooft loop operators are classified by elements of $\Lambda_{cw}(G)/\mathcal{W}$. We will denote the 't Hooft operator corresponding to the coweight μ as T_μ.

By definition, $\Lambda_{cw}(G)$ is identified with the weight lattice $\Lambda_w({}^LG)$ of ${}^LG$. But elements of $\Lambda_w({}^LG)/\mathcal{W}$ are in one-to-one correspondence with irreducible representations of ${}^LG$. This suggests that MO duality maps the 't Hooft operator corresponding to a coweight $\mu \in \Lambda_{cw}(G)$ to the Wilson operator corresponding to a representation ${}^LR$ with highest weight in the Weyl orbit of $\mu \in \Lambda_w({}^LG)$. This is a reinterpretation of the the Goddard-Nuyts-Olive argument discussed in section 2 in terms of operators rather than states.

To test this duality, one can compare the algebra of 't Hooft operators for gauge group G and Wilson operators for gauge group ${}^LG$. In the latter case, the operator product is controlled by the algebra of irreducible representations of ${}^LG$. That is, we expect that as the loop γ' approaches γ, we have

$$W_R(\gamma)W_{R'}(\gamma') \sim \bigoplus_{R_i \subset R\otimes R'} W_{R_i}(\gamma),$$

where R and R' are irreducible representations of G, and the sum on the right-hand-side runs over irreducible summands of $R \otimes R'$.

In the case of 't Hooft operators the computation of the operator product is much more nontrivial [6]. We will only sketch the procedure and state the results. First, one considers the twisted YM theory (at $t = 1$) on a 4-manifold of the form $X = \mathbb{R} \times I \times C$, where I is an interval and $\mathbb{R}$ is regarded as the time direction. The computation is local, so one may even take $C = \mathbb{P}^1$. The 't Hooft operators are located at points on $I \times C$ and extend in the time direction. Their presence modifies the definition and Hamiltonian quantization of the gauge theory on X. Namely, the gauge field A and the scalar ϕ_0 have prescribed singularities at points on $I \times C$. Since the theory is topological, one can take the limit when the volume of $C \times I$ goes to zero; in this limit the theory reduces to a 1d theory: supersymmetric sigma-model on $\mathbb{R}$ whose target is the space of vacua of the YM theory. The latter space can be obtained by solving BPS equations (5) assuming that all fields are independent of the time coordinates. With suitable boundary conditions, one can show that this moduli space is the space of solutions of Bogomolny equations on $I \times C$ with prescribed singularities.

The Bogomolny equations are equations for a gauge field A and a Higgs field $\phi_0 \in \mathrm{ad}(E)$ on a 3-manifold Y (which in our case is $C \times I$):

$$F = \star_3 d_A \phi_0.$$

To understand the moduli space of solutions of this equation, let us rewrite it as an evolution equation along I. Letting σ to be a coordinate on I, and working in the gauge $A_\sigma = 0$, we get

$$\partial_\sigma A_{\bar{z}} = -i D_{\bar{z}} \phi_0.$$

This equation says that the isomorphism class of the holomorphic G-bundle on $C \times y$, $y \in I$ corresponding to $A_{\bar{z}}$ is independent of y. This conclusion is violated only at the points on I where the 't Hooft operators are inserted. A further analysis shows that at these points the holomorphic G-bundle undergoes a Hecke transformation.

By definition, Hecke transformations modify the G-bundle at a single point. The space of such modifications is parameterized by points of the affine Grassmannian $Gr_G = G((z))/G[[z]]$. This is an infinite-dimensional space which is a union of finite-dimensional strata called Schubert cells [40]. Schubert cells are labeled by Weyl-equivalence classes of coweights of G. As explained in [6], the Hecke transformations corresponding to an 't Hooft operator T_μ are precisely those in the Schubert cell labeled by μ.

The net result of this analysis is that for a single 't Hooft operator T_μ the space of solutions of the BPS equations is the Schubert cell $\mathcal{C}_\mu$. The Hilbert space of the associated 1d sigma-model is the L^2 cohomology of $\mathcal{C}_\mu$. More generally, computing the product of 't Hooft operators reduces to the study of L^2 cohomology of the Schubert cells. Assuming that the L^2 cohomology coincides with the intersection cohomology of the closure of the cell, the prediction of the MO duality reduces to the statement of the geometric Satake correspondence, which says that the tensor category of equivariant perverse sheaves on Gr_G is equivalent to the category of finite-dimensional representations of ${}^L G$ [41, 42, 43]. This provides a new and highly nontrivial check of the MO duality.

From the gauge-theoretic viewpoint one can think about 't Hooft operators as functors from the category of A-branes on $\mathcal{M}_H(G, C)$ to itself. To understand how this comes about, consider a loop operator (Wilson or 't Hooft) in the twisted $\mathcal{N} = 4$ SYM theory (at $t = i$ or $t = 1$, respectively). As usual, we take the four-manifold X to be $\Sigma \times C$, and let the curve γ be of the form $\gamma_0 \times p$, where $p \in C$ and γ_0 is a curve on Σ. Let $\partial\Sigma_0$ be a connected component of $\partial\Sigma$ on which we specify a boundary condition corresponding to a given brane β. This brane is either a B-brane in complex structure J or an A-brane in complex structure K, depending on whether $t = i$ or $t = 1$. Now suppose γ_0 approaches $\partial\Sigma_0$. The "composite" of $\partial\Sigma_0$ with boundary condition β and the loop operator can be thought of as a new boundary condition for the topological sigma-model with target $\mathcal{M}_H(G, C)$. It depends on $p \in C$ as well as other data defining the loop operator. One can show that this "fusion" operation defines a functor from the category of topological branes to itself [6].

In the case of the Wilson loop, it is very easy to describe this functor. In complex structure J, we can identify $\mathcal{M}_H(G, C)$ with $\mathcal{M}_{flat}(G, C)$. On the product $\mathcal{M}_{flat}(G, C) \times C$ there is a universal G-bundle which we call $\mathcal{E}$. For any $p \in C$ let us denote by $\mathcal{E}_p$ the restriction of $\mathcal{E}$ to $\mathcal{M}_{flat}(G, C) \times p$. For any representation R of G we can consider the operation of tensoring coherent sheaves on $\mathcal{M}_{flat}(G, C)$ with the associated holomorphic vector bundle $R(\mathcal{E}_p)$. One can show that this is the functor corresponding to the Wilson loop in representation R inserted at a point $p \in C$. We will denote this functor $W_R(p)$. The action of 't Hooft loop operators is harder to describe, see sections 9 and 10 of [6] for details. In particular, it is shown there that 't Hooft operators act by Hecke transformations.

Consider now the structure sheaf $\mathcal{O}_x$ of a point $x \in \mathcal{M}_{flat}({}^LG, C)$. For any representation ${}^LR$ of ${}^LG$ the functor corresponding to $W_{{}^LR}(p)$ maps $\mathcal{O}_x$ to the sheaf $\mathcal{O}_x \otimes {}^LR(\mathcal{E}_p)_x$. That is, $\mathcal{O}_x$ is simply tensored with a vector space ${}^LR(\mathcal{E}_p)_x$. One says that $\mathcal{O}_x$ is an eigenobject of the functor $W_{{}^LR}(\mathcal{E}_p)$ with eigenvalue ${}^LR(\mathcal{E}_p)_x$. The notion of an eigenobject of a functor is a categorification of the notion of an eigenvector of a linear operator: instead of an element of a vector space one has an object of a $\mathbb{C}$-linear category, instead of a linear operator one has a functor from the category to itself, and instead of a complex number (eigenvalue) one has a complex vector space ${}^LR(\mathcal{E}_p)_x$.

We conclude that $\mathcal{O}_x$ is a common eigenobject of all functors $W_{{}^LR}(p)$ with eigenvalues ${}^LR(\mathcal{E}_p)_x$. Actually, since we can vary p continuously on C and the vector spaces ${}^LR(\mathcal{E}_p)_x$ are naturally identified as one varies p along any path on C, it is better to say that the eigenvalue is a flat ${}^LG$-bundle ${}^LR(\mathcal{E})_x$. Tautologically, this flat vector bundle is obtained by taking the flat principal ${}^LG$-bundle on C corresponding to x and associating to it a flat vector bundle via the representation ${}^LR$.

Applying the MO duality, we may conclude that the A-brane on $\mathcal{M}_H(G, C)$

corresponding to a fiber of the Hitchin fibration is a common eigenobject for all 't Hooft operators, regarded as functors on the category of A-branes. The eigenvalue is the flat ${}^LG$ bundle on C determined by the mirror of the A-brane. This is the gauge-theoretic version of the statement that the D-module on $\mathrm{Bun}_G(C)$ corresponding to a point on $\mathcal{M}_{flat}({}^LG, C)$ is a Hecke eigensheaf.

9 Quantum geometric Langlands duality

One possible generalization of geometric Langlands duality is to consider twisted Yang-Mills theory with $\theta \neq 0$. Making θ nonzero in the gauge theory corresponds to turning on a topologically nontrivial B-field in the corresponding topological sigma-model on Σ:

$$B = -\frac{\theta}{2\pi}\omega_I.$$

At $t = i$ this deformation does not affect the topological sigma-model, because one can make it a 2-form of type $(2,0)$ by adding an exact form, and $(2,0)$ B-fields correspond to BRST-exact deformations of the action. Equivalently, one notes that all dependence on θ in the gauge theory is through the parameter Ψ, and for $t = i$ $\Psi = \infty$ irrespective of the value of θ.

On the other hand, for $t = 1$ the deformation has a nontrivial effect, as it makes Ψ real (for $t = 1$ and $\theta = 0$ we have $\Psi = 0$). Of course, there is no paradox here: turning on θ at $t = 1$ does not correspond to turning on θ at $t = i$. To understand the implications of a nonzero θ at $t = 1$, note that keeping $t = 1$ and varying θ we can get arbitrary real values of Ψ. On the other hand, duality maps $\Psi \to -1/(n_{\mathfrak{g}}\Psi)$. Thus one can say that MO duality maps the twisted YM theory with $t = 1$ and a nonzero $\theta = \theta_0$ to a twisted YM theory with $t = 1$ and $\theta = -4\pi^2/\theta_0$. That is, it maps an A-model in complex structure K for $\mathcal{M}_H(G, C)$ to an A-model in complex structure K for $\mathcal{M}_H({}^LG, C)$.

To understand the mathematical implications of this statement, we need to reinterpret the category of A-branes on $\mathcal{M}_H(G, C)$ when the B-field is proportional to ω_I. Again, the key insight is that for any such B-field there exists an analogue of the canonical coisotropic A-brane, such that the corresponding sheaf of open string states is isomorphic to the sheaf of twisted differential operators on $\mathcal{M}(G, C)$.

The curvature F of the line bundle on a coisotropic A-brane should satisfy

$$\left(\omega^{-1}(F+B)\right)^2 = -1,$$

where $\omega = \mathrm{Im}\tau\,\omega_K$. We take

$$F = \mathrm{Im}\tau \cdot \cos q, \quad \sin q = -\frac{\mathrm{Re}\tau}{\mathrm{Im}\tau}$$

as the curvature of the canonical coisotorpic A-brane for nonzero θ. The corresponding sheaf of open strings, regarded as a sheaf of vector spaces, is isomorphic to the sheaf of functions on $\mathcal{M}_H(G,C)$ holomorphic in the complex structure

$$I(\Psi) = I\cos q - J\sin q.$$

The corresponding holomorphic coordinates on $\mathcal{M}_H(G,C)$ are

$$A'_{\bar{z}} = A_{\bar{z}} - i\tan\frac{q}{2}\phi_{\bar{z}}, \quad \phi'_z = \phi_z + i\tan\frac{q}{2}A_z.$$

For $q=0$, the complex structure $I(\Psi)$ is simply I, and the corresponding complex manifold is birational to the cotangent bundle to the moduli space of holomorphic G-bundles on C. For $q \neq 0$ one can show that $\mathcal{M}_H(G,C)$ is birational to the space of pairs (E, ∂_λ), where E is a holomorphic principal G-bundle on C and ∂_λ is a λ-connection on E, with $\lambda = i\tan\frac{q}{2}$. We remind that a λ-connection on a holomorphic G-bundle E is a map

$$\partial_\lambda : \Gamma(E) \to \Gamma(E \otimes \Omega^1)$$

such that

$$\partial_\lambda(fs) = f\partial_\lambda s + \lambda df \otimes s, \quad \forall f \in \Gamma(\Omega^0), s \in \Gamma(E).$$

The space of λ-connections on E is an affine space modeled on the space of Higgs fields on E. Thus in complex structure $I(\Psi)$ the space $\mathcal{M}_H(G,C)$ is birational to an affine bundle modeled on the cotangent bundle of $\mathcal{M}(G,C)$. We will call this affine bundle the twisted cotangent bundle.

The rest of the argument proceeds much in the same way as in the case $\theta = 0$. On the classical level, the sheaf of boundary observables corresponding to the canonical coisotropic A-brane is quasiisomorphic to the sheaf of holomorphic functions on the twisted cotangent bundle over $\mathcal{M}(G,C)$. On the quantum level, the algebra of boundary observables becomes noncommutative, and to ensure that there are no problems with the definition of the product one needs to work with functions which are polynomial in the fiber coordinates. Thus we end up with a sheaf on $\mathcal{M}(G,C)$ which locally looks like the sheaf of symbols of holomorphic differential operators on $\mathcal{M}(G,C)$. The action of the A-model now has the form

$$S = \int_\Sigma \Phi^*(\omega - iF - iB) = -i\int_\Sigma \Phi^*\Omega'$$

where the holomorphic symplectic form Ω' on the space of λ-connections is given by

$$\Omega' = -\frac{\operatorname{Im}\tau}{\pi}\int_C \operatorname{Tr}\left(\delta\phi_z \delta A'_{\bar z}\right) + i\frac{\operatorname{Re}\tau}{\pi}\int_C \operatorname{Tr}\left(\delta A'_z \delta A'_{\bar z}\right).$$

The first term in this formula is exact, but the second one is not. It is proportional to the pull-back of the curvature of the determinant line bundle on $\mathcal{M}(G,C)$. Because of this, the action cannot be written as a boundary term globally on $\mathcal{M}(G,C)$. But locally the form Ω' is still exact, and therefore we end up with the problem of quantizing the canonical commutation relations

$$[p_i, q^j] = \operatorname{Im}\tau\, \delta_i^j, \quad [p_i, p_j] = [q_i, q_j] = 0.$$

The quantization is unique locally and gives holomorphic differential operators on some power of a line bundle $\mathcal{L}$ over $\mathcal{M}(G,C)$. Since $H^2(\mathcal{M}(G,C))$ and generated by the first Chern class of the determinant line bundle Det, we may parameterize this complex power of a line bundle as $K \otimes \mathrm{Det}^q$, $q \in \mathbb{C}$. We conclude that in the case $\theta \neq 0$ the sheaf of boundary observables on the canonical coisotropic A-brane is quasiisomorphic to the sheaf of holomorphic differential operators on $K \otimes \mathrm{Det}^q$ for some complex number q.

It remains to fix q as a function of θ. For $\theta \neq 0$ we cannot appeal to time-reversal symmetry since nonzero θ breaks it. Instead, we note that for $\theta = 2\pi n$ we must have $q = n$. Indeed, for these values of q we may consider the locus $\phi = 0$ in $\mathcal{M}(G,C)$ equipped with the gauge field $F = -n\omega_I$ as a Lagrangian A-brane (recall that for nonzero B-field the gauge field on the Lagrangian brane is not flat but satisfies F+B=0). It is also easy to show that the sheaf of open strings which begin on the canonical coisotropic A-brane and end on this Lagrangian A-brane is $K \otimes \mathrm{Det}^n$ (this follows from the fact that ω_I is the curvature of Det). Finally, one can show that q is linear in θ. Therefore we must have $q = \theta/(2\pi) = \Psi$ for all Ψ.

In the same way as before, by considering the sheaf of open strings beginning on the c.c. brane and ending on any other A-brane, we can assign to any A-brane a twisted D-module, i.e. a sheaf of modules over the sheaf of holomorphic differential operators on $K \otimes \mathrm{Det}^{\Psi}$. Conjecturally, this map can be extended to an equivalence of categories. Therefore for $\Psi \neq 0$ the Montonen-Olive duality would say that this category would remain unchanged if we replaced G by ${}^{L}G$ and Ψ by $-1/(n_{\mathfrak{g}}\Psi)$. This is called quantum geometric Langlands duality. The name "quantum" comes from the fact that now on both sides of the duality we have to deal with modules over noncommutative algebras. Unlike in the "classical" case, here there is nothing comparable to the Hecke eigensheaf. Physically, the reason for this is that the fibers of the Hitchin fibration are not valid A-branes for nonzero θ.

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