Geometry of Soergel Bimodules

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(joint with Geordie Williamson)

IAS/MIT

June 17th, 2007

Outline

References:

This slide show can be downloaded from

http://math.berkeley.edu/~bwebste/HH-SB.pdf

Some references:

- B. W. and G. W., *A geometric model for the Hochschild homology of Soergel bimodules*.
  (**http://math.berkeley.edu/ bwebste/hochschild-soergel.pdf**)
- W. Soergel, *Kategorie O, Perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*.
- W. Soergel, *The combinatorics of Harish-Chandra bimodules*.
- M. Khovanov, *Triply-graded link homology and Hochschild homology of Soergel bimodules*.
- J. Bernstein and V. Lunts, *Equivariant sheaves and functors*.

Soergel bimodules

Let $R = \mathbb{C}[x_1, \ldots, x_n]/(x_1 + \cdots + x_n)$, and s_i be the map permuting x_i and x_{i+1} and let $G = \mathrm{SL}(n, \mathbb{C})$.

Like so many objects in mathematics, Soergel bimodules have a number of definitions:

1. One which explains why anyone ever cared:

Definition

A **Soergel bimodule** *is the image of a projective object in category* $\tilde{\mathcal{O}}$ *under Soergel's "combinatoric" functor* $\mathbb{V}$.

2. One which is hands-on but totally unilluminating:
3. One which involves disgusting levels of machinery, but which ultimately is the best for working with:

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A **Soergel bimodule** *is a direct sum of summands of tensor products*

$$R \otimes_{R^{s_{i_1}}} R \otimes_{R^{s_{i_2}}} \cdots \otimes_{R^{s_{i_m}}} R$$

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A **Soergel bimodule** *is the hypercohomology of a semi-simple $B \times B$-equivariant perverse sheaf on G.*

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3. One which involves disgusting levels of machinery, but which ultimately is the best for working with: *perverse sheaves*

While intimidating at first, a multiplicity of definitions is, in fact, a strength rather than a weakness, allowing us to our problems translate back and forth at will.

Soergel bimodules for $n = 2$

When $n = 2$, then

$$R = \mathbb{C}[x_1, x_2]/(x_1 + x_2) \cong \mathbb{C}[y]$$

with the action of s_1 sending $y \mapsto -y$. Thus, $R^{s_1} = \mathbb{C}[y^2]$ and

$$R_1 \cong R \otimes_{R^{s_1}} R \cong \mathbb{C}[y \otimes 1, 1 \otimes y] \cdot r/(y^2 \otimes 1 - 1 \otimes y^2)$$

Proposition

The elements r and $(1 \otimes y - y \otimes 1) \cdot r$ generate $R_1 \otimes_R R_1$ as an R-bimodule, and generate two summands, so $R_1 \otimes_R R_1 \cong R_1 \oplus R_1\{2\}$.

Corollary

Every indecomposable Soergel bimodule for $n = 2$ is isomorphic to R or R_1.

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When $n = 3$, similar calculations show

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Every indecomposable Soergel bimodule for $n = 2$ is isomorphic to one of $R, R_1, R_2, R_1 \otimes_R R_2, R_2 \otimes_R R_1$ or $R \otimes_{R^{S_3}} R$.

Anyone used to playing with SL(3) will probably note that we have an obvious bijection from S_3 to the set of indecomposable Soergel bimodules:

$$\begin{array}{ccc} 1 \leftrightarrow R & (12) \leftrightarrow R_1 & (23) \leftrightarrow R_2 \\ (123) \leftrightarrow R_2 \otimes_R R_1 & (132) \leftrightarrow R_1 \otimes_R R_2 & (13) \leftrightarrow R \otimes_{R^{S_3}} R \end{array}$$

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Indecomposable Soergel bimodules

Question

In general, is the set of indecomposable Soergel bimodules in bijection with S_n?

Definition 2 is perfectly useless at answering this sort of question. But from the perspectives of Definitions 1 or 3, it borders on obvious:

Proposition (Soergel)

Every indecomposable Soergel bimodule is of the form

$$R_w = IH^*_{B\times B}(\overline{BwB}) = \mathbb{H}^*_{B\times B}(\mathbf{IC}(\overline{BwB})),$$ What?

for $w \in S_n$ (and these are pairwise not isomorphic).

Let $G_w = \overline{BwB}$.

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Bott-Samelson Soergel bimodules

Since they appear in the definition of Khovanov-Rozansky homology, we will also be interested in the **Bott-Samelson bimodules**

$$R_{\mathbf{i}} \cong R \otimes_{R^{s_{i_1}}} \cdots \otimes_{R^{s_{i_m}}} R \cong R_{s_{i_1}} \otimes_R \cdots \otimes_R R_{s_{i_m}}$$

In the formalism that Rasmussen and Rozansky have used, this is the bimodule corresponding to a singular braid diagram.

If $\mathbf{i}' = \{i_1, \ldots, \hat{i}_k, \ldots, i_m\}$, then the modules $R_{\mathbf{i}}$ and $R_{\mathbf{i}'}$ have natural maps $\pi_k : R_{\mathbf{i}}\{1\} \to R_{\mathbf{i}'}$ and $\rho_k : R_{\mathbf{i}'}\{1\} \to R_{\mathbf{i}}$.

All differentials in the Rouquier complex are built from these natural maps.

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Bott-Samelson spaces

Let P_i the parabolic preserving the standard flag minus its i-dimensional subspace, and let

$$G_{\mathbf{i}} \cong P_{i_1} \times_B P_{i_2} \times_B \cdots \times_B P_{i_m}.$$

This space is smooth, and has a $B \times B$-action by left and right multiplication.

Proposition

For all $\mathbf{i} = (i_1, \ldots, i_m)$, we have $R_{\mathbf{i}} \cong H^*_{B\times B}(G_{\mathbf{i}})$.

We have a natural inclusion $G_{\mathbf{i}'} \subset G_{\mathbf{i}}$, and the maps ρ_k, π_k are simply pushforward and pullback in cohomology.

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Hochschild homology

Hochschild homology also naturally appears in the construction of KR homology. It is the derived functor of $HH^0(M) = M/[R,M]$ where as usual

$$[R,M] = R \cdot \{r \cdot m - m \cdot r | r \in R, m \in M\} \cdot R \subset M.$$

Definition

For any projective resolution of M,

$$\mathbf{P}^\bullet = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

$HH^*(M)$ is the homology of the complex $HH^0(\mathbf{P}^\bullet)$.

Note that $HH^*(R_w)$ has the obvious "Hochschild" grading (which is independent of any grading on R) and another "polynomial" grading which arises from using a graded projective resolution of R_w.

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Hochschild homology as extension of scalars

Consider an $R-R$ bimodule M as an $R \otimes R$ module. Note that the R-module $M/[R,M]$ can be rewritten as the extension of scalars

$$M/[R,M] \cong M \otimes_{R \otimes R} R$$

By the standard yoga of homological algebra, Hochschild homology can be reinterpreted as a derived extension of scalars.

$$HH^*(M) \cong M \overset{L}{\otimes}_{R \otimes R} R$$

Hochschild homology can thus be interpreted geometrically using Bernstein and Lunts's equivalence between the equivariant derived category $D_T(pt)$ with dg-modules over $H_T^*(pt) \cong R$.

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The geometry of Hochschild homology

While the general story has some surprising subtleties, we get a rather simple answer for Soergel bimodules.

Proposition (W.-Williamson)

$$HH^*(R_w) \cong IH^*_B(G_w) \cong IH^*_T(G_w) \qquad HH^*(R_{\mathbf{i}}) \cong H^*_B(G_{\mathbf{i}}) \cong H^*_T(G_{\mathbf{i}})$$

where B, T acts on $G_w, G_{\mathbf{i}}$ by conjugation. This isomorphism is "functorial", i.e. for any map φ of $B \times B$-sheaves, we have

$$HH^*(\mathbb{H}^*_{B\times B}(\varphi)) = \mathbb{H}^*_B(\varphi)$$

and takes the natural grading on cohomology to the "polynomial" grading on HH^* minus the "Hochschild" grading.

Keep in mind that if G_w is smooth, then $IH^*_B(G_w) \cong H^*_B(G_w)$.

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Keep in mind that if G_w is smooth, then $IH_B^*(G_w) \cong H_B^*(G_w)$.

Equivariant formality

Equivariant cohomology is easiest to understand in the presence of equivariant formality.

Definition

We say a T-space X is **equivariantly formal** *if one of the following equivalent conditions holds*

1. $H_T^*(X)$ *is free as a module over R.*
2. $H_T^*(X) \cong R \otimes_{\mathbb{C}} H^*(X)$.
3. $\dim_{\mathbb{C}} H^*(X^T) = \dim_{\mathbb{C}} H^*(X)$.
4. $\dim_{\mathbb{C}} H^*(X^T) \geq \dim_{\mathbb{C}} H^*(X)$.

Theorem

If X is equivariantly formal, then the pullback map $H_T^*(X) \to H_T^*(X^T)$ is injective and is an isomorphism after tensoring with $Q = (R^\times)^{-1}R$.

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Equivariant formality of $G_{\mathbf{i}}$

Theorem (Rasmussen, W.-Williamson)

The T-spaces $G_{\mathbf{i}}$ equivariantly formal with respect to the conjugation T-action for all $\mathbf{i}$. Equivalently, the sheaves $\mathbf{IC}(G_w)$ are equivariantly formal for all w.

Proof: Rasmussen proved algebraically that $HH^*(R_{\mathbf{i}})$ is free. By definition 1, this implies equivariant formality.

Corollary

For all w, we have $HH^*(R_{\mathbf{i}}) \cong R \otimes H^*(G_{\mathbf{i}})$.

Applying the Hirsch lemma to the fibration $G_{\mathbf{i}} \to G_{\mathbf{i}}/B$, and taking the Euler characteristic of the resulting complex shows Rasmussen's results indentifying a specialization of the Hilbert series of $HH^*(R_{\mathbf{i}})$ with the Hilbert series of $H^*(G_{\mathbf{i}}/B)$.

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Proof: Rasmussen proved algebraically that $HH^*(R_{\mathbf{i}})$ is free. By definition 1, this implies equivariant formality.

Corollary

For all w, we have $HH^*(R_{\mathbf{i}}) \cong R \otimes H^*(G_{\mathbf{i}})$.

Applying the Hirsch lemma to the fibration $G_{\mathbf{i}} \to G_{\mathbf{i}}/B$, and taking the Euler characteristic of the resulting complex shows Rasmussen's results indentifying a specialization of the Hilbert series of $HH^*(R_{\mathbf{i}})$ with the Hilbert series of $H^*(G_{\mathbf{i}}/B)$.

Gradings

Since $G_{\mathbf{i}}$ is smooth and equivariantly formal, we can separate the Hochschild and polynomial gradings.

Recall that by the Künneth theorem, $H_T^*(G_{\mathbf{i}}^T) \cong R \otimes_{\mathbb{C}} H^*(G_{\mathbf{i}})$, so we can write the usual grading as a sum of "equivariant" and "topological" gradings.

By the equivariant formality, the pullback map $H_T^*(G_{\mathbf{i}}) \to H_T^*(G_{\mathbf{i}}^T)$ is injective. Using the above splitting, we can give $H_T^*(G_{\mathbf{i}})$ a similar bigrading.

Proposition

The isomorphism $H_T^*(G_{\mathbf{i}}) \cong HH^*(R_{\mathbf{i}})$ takes the "topological" to the "Hochschild" grading and the "equivariant" to the "polynomial" minus twice the "Hochschild."

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The Hochschild homology of smooth Soergel bimodules

Also, we can use this theorem to describe the Hochschild homology of smooth Soergel bimodules. Assume G_w is smooth.

Proposition (W.-Williamson)

We have an isomorphism

$$HH^*(R_w) \cong R \otimes_{\mathbb{C}} H^*(G_w) \cong R \otimes_{\mathbb{C}} \wedge^\bullet(\gamma_1, \dots, \gamma_{n-1})$$

where the # of indices i with $\deg(\gamma_i) = (2m, 1)$ is the number of positive roots α with $w(\alpha)$ negative with $\langle \alpha, \rho \rangle = m - 1$ minus the number with $\langle \alpha, \rho \rangle = m$.

It is worth noting that $w \in S_n$ with G_w smooth are characterized combinatorially by pattern avoidance, and as the varieties defined by non-crossing inclusions.

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The structure of $G_{\mathbf{i}}$

Of course, the Holy Grail of this business is really properly understanding $H^*_T(G_{\mathbf{i}})$. Since this space is equivariantly formal, understanding $H^*(G_{\mathbf{i}})$ and $H^*(G_{\mathbf{i}}^T)$ would be a good start.

Proposition

The fixed points of the conjugation T action on $K_{\mathbf{i}}$ is the subset

$$G_{\mathbf{i}}^T = \bigsqcup_{\epsilon_i \in \{0,1\}; s_1^{\epsilon_1} \cdots s_m^{\epsilon_m} = e} T \cdot (s_1^{\epsilon_1}, \dots, s_m^{\epsilon_m})$$

So $H^*(K_{\mathbf{i}}^T)$ is just a number of copies of $H^*(T)$.

Question

What is $H^*(G_{\mathbf{i}})$? We know that $\dim_{\mathbb{C}} H^*(G_{\mathbf{i}}) = \dim_{\mathbb{C}} H^*(G_{\mathbf{i}}^T)$ and not much else.

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The pullback morphism $H_T^*(G_{\mathbf{i}}) \to H_T^*(G_{\mathbf{i}}^T)$

From geometry, we get a map

$$\pi_{\mathbf{s}} : R_{\mathbf{i}} \cong H^*_{T\times T}(G_{\mathbf{i}}) \to H^*_{T\times T}(T \cdot (s_1^{\epsilon_1}, \dots, s_m^{\epsilon_m})) \cong R$$

for each sequence $\mathbf{s}$, such that $s_1^{\epsilon_1} \cdots s_m^{\epsilon_m} = e$.

This can be defined algebraically by $\pi_{i_1,\epsilon_1} \otimes \cdots \otimes \pi_{i_m,\epsilon_m}$ where $\pi_{i_j,\epsilon_j} : R_i \to R(s_i^\epsilon)$ (here $R(s_i^\epsilon)$ is R with the right action twisted by s_i^ϵ) is the map $\pi_{i,\epsilon}(a \otimes b) = a(b^{s_i^\epsilon})$.

By equivariant formality, the map $HH^*(\bigoplus_{\mathbf{s}} \pi_{\mathbf{s}})$ is injective and an isomorphism after tensoring with the fraction field Q.

What's left

Summary: our model

- gives a uniform description of the Hochschild homology of indecomposable and Bott-Samelson Soergel bimodules.
- allows us to compute certain cases, as well as leverage for understanding general properties of this homology.
- gives a geometric description of the Rouquier complex.

What we hope for is

- a better understanding of the Bott-Samelson space
- a better understanding of non-smooth cases.
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Equivariant cohomology

Let EG be a contractible space on which G acts freely.

Definition

The equivariant cohomology $H^*_G(X) = \mathbb{H}^*_G(\underline{\mathbb{C}}_X)$ of a G-space X is the cohomology of the Borel space $EG \times_G X$. In particular, $H^*_G(pt) \cong H^*(BG)$ where $BG = EG/G$.

There is a variant of equivariant cohomology called **equivariant intersection cohomology** $IH^*_G(X) = \mathbb{H}^*_G(\mathbf{IC}_X)$ which is better suited for singular spaces, but is the same as $H^*_G(X)$ for smooth spaces.

We have a map $EG \times_G X \to BG$, giving us an action of $H^*_G(pt)$ on $IH^*_G(X)$.

Proposition

We have a natural isomorphism $H^*_B(pt) \cong R$, so $H^*_{B\times B}(pt) \cong R \otimes R$. This geometric action makes R_w into an $R - R$-bimodule. Back to Soergel-land.

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