Towards an algebro-geometric proof of the Razumov–Stroganov conjecture?

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June 20, 2007

Outline of the talk

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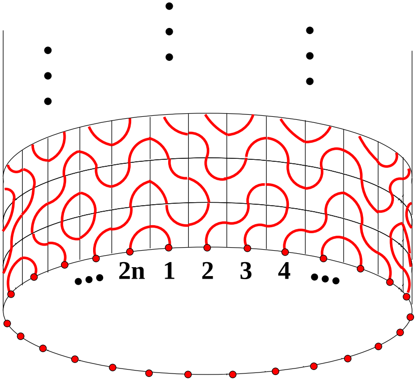
Consider the following probabilistic model. Fill some two-dimensional surface with boundary with plaquettes:

 with probability p,  with probability $1 - p$. $(0 < p < 1)$



Case of the half-infinite cylinder geometry ("periodic boundary conditions")

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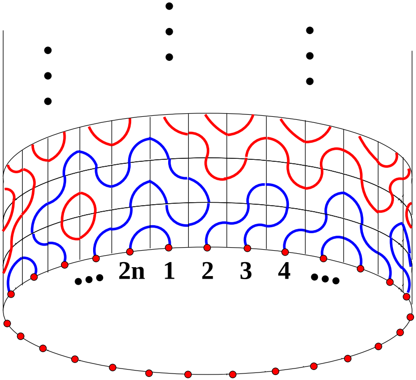


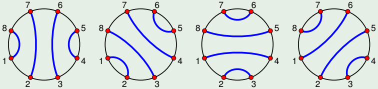
Case of the half-infinite cylinder geometry ("periodic boundary conditions")

Probability law of the connectivity of the external vertices?

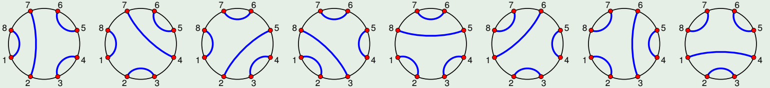
The connectivity of the external vertices can be encoded into a link pattern = a planar pairing of $2n$ points on a circle.

Example

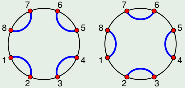
In size $L = 2n = 8$,



$\frac{1}{42}$



$\frac{3}{42}$



$\frac{7}{42}$

Observations (de Gier, Nienhuis '01)

Define

$$A_n = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} = 1, 2, 7, 42, 429\ldots$$

Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type

, are 1:

1. All components are Ψ are (positive) integers. (Di Francesco, PZJ '07)
2. The largest components of Ψ correspond to patterns of the type

 and are equal to A_{n-1}.
(Di Francesco, PZJ + Zeilberger '07 *or* Razumov, Stroganov, PZJ '07)
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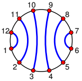
Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type 

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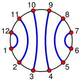
Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type 

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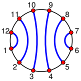
Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type 

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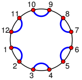
2. The largest components of Ψ correspond to patterns of the type 

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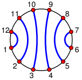
Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type 

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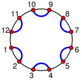
2. The largest components of Ψ correspond to patterns of the type 

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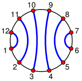
Form the vector Ψ of unnormalized probabilities, so that the smallest components, with patterns of the type 

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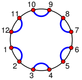
2. The largest components of Ψ correspond to patterns of the type 

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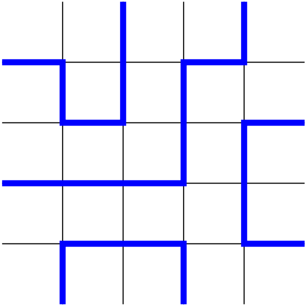
Example

Fully Packed Loops

A Fully Packed Loop configuration (FPL) on a $n \times n$ square grid:



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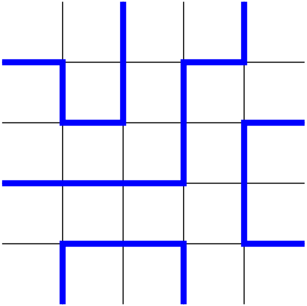

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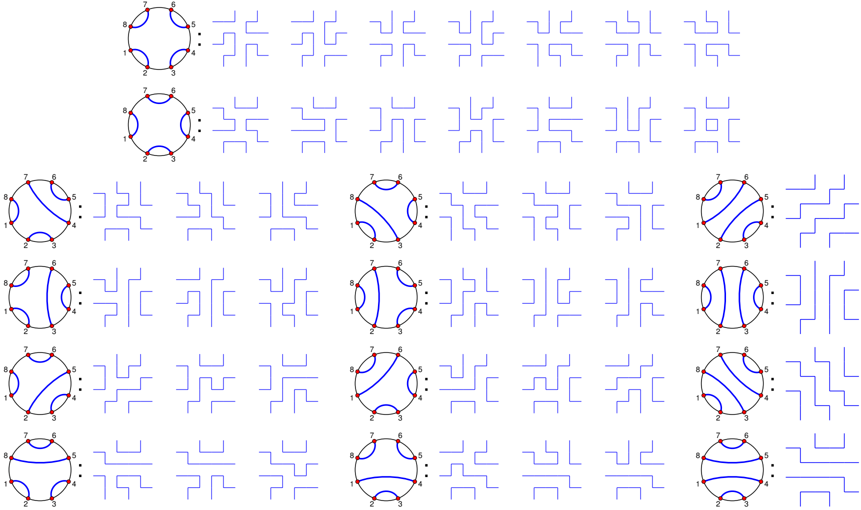
The number of FPLs (a.k.a. ASMs) of size n is A_n.

It is natural to group FPLs by connectivity of their endpoints: cf



Razumov–Stroganov conjecture

Conjecture (Razumov, Stroganov '01)

Denote by $A(\pi)$ the number of FPLs with connectivity described the link pattern π. This is exactly the (unnormalized) probability of pattern π in the Temperley–Lieb model of loops.

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Introduction of inhomogeneities into the loop model

Consider the same probabilistic model but with probabilities p_i depending on the column $i = 1, \ldots, 2n$:

 $: p_i = \dfrac{q\, z_i - q^{-1} t}{q\, t - q^{-1} z_i}$  $: 1 - p_i = \dfrac{z_i - t}{q\, t - q^{-1} z_i}$

with $q = e^{2i\pi/3}$.

z_i are the spectral parameters.

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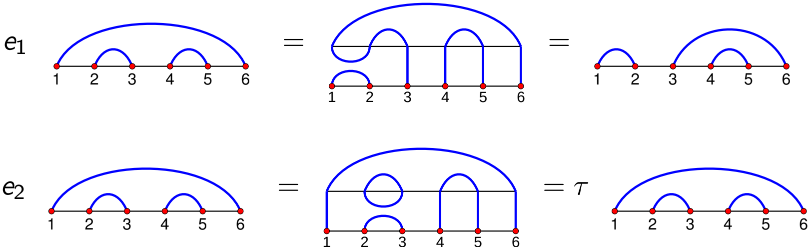
Temperley–Lieb algebra

The Temperley–Lieb algebra $\mathrm{TL}_L(\tau)$ (a quotient of the Hecke algebra) is defined by generators e_i, $i = 1, \ldots, L-1$, and relations

$$e_i^2 = \tau e_i \qquad e_i e_{i\pm1} e_i = e_i \qquad e_i e_j = e_j e_i \qquad |i-j| > 1$$

Define the action of Temperley–Lieb generators e_i on link patterns:



where the weight of a closed loop is τ.

R-matrix

Set $\tau = -q - 1/q$ (here q is generic), and define the R-matrix:

$$\check{R}_i(u) = \frac{(q\,u - q^{-1})I + (u-1)e_i}{q - q^{-1}u}$$

It satisfies the Yang–Baxter equation:

$$\check{R}_i(u)\check{R}_{i+1}(uv)\check{R}_i(v) = \check{R}_{i+1}(v)\check{R}_i(uv)\check{R}_{i+1}(u)$$

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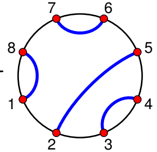
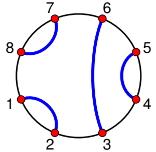
Smirnov's qKZ system

Consider the following system of equations for Ψ, a vector-valued polynomial in $z_1, \ldots, z_L, q, q^{-1}$: $(i = 1, \ldots, L-1)$

$$\check{R}_i(z_{i+1}/z_i)\Psi(z_1, \ldots, z_L) = \Psi(z_1, \ldots, z_{i+1}, z_i, \ldots, z_L) \quad (1)$$

$$\sigma^{-1}\Psi(z_1, \ldots, z_L) = c\,\Psi(z_2, \ldots, z_L, s\,z_1) \quad (2)$$

where σ rotates link patterns:

σ  $=$ 

Level 1 Polynomial solution of qKZ

Fact

In size $L = 2n$, for $s = q^6$ (level 1), there exists a polynomial solution of degree $n(n-1)$, unique up to normalization.

Example ($L = 2n = 4$)

Ψ

$= (q\, z_1 - q^{-1} z_2)(q\, z_3 - q^{-1} z_4)$

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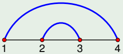
Ψ 

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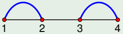
Ψ 

$= (q\, z_2 - q^{-1} z_3)(q^{-2} z_4 - q^2 z_1)$

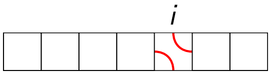
qKZ equation à la Frenkel–Reshetikhin

The actual qKZ equation is a consequence of (1) and (2):

$$\Psi(z_1, \ldots, s\, z_i, \ldots, z_L) = S_i(z_1, \ldots, z_{2n}) \Psi(z_1, \ldots, z_i, \ldots, z_L)$$

$(i = 1, \ldots, L-1)$ where

$$S_i(z_1, \ldots, z_{2n}) =$$



Remark: qKZ is a system of compatible difference equations, in the same way that KZ is a system of compatible differential equations.

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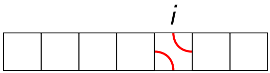

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Special point $q^3 = 1$

Assume $q = e^{\pm 2i\pi/3}$. Then $s = 1$ (rotational invariance is restored) and one can show that Ψ is the vector of (unnormalized) probabilities of the inhomogeneous loop model. The homogeneous case is recovered when $z_i = 1$.

Homogeneous limit for generic q

Remark: Keeping q generic, one can consider the homogeneous limit $z_i = 1$.

Example ($L = 2n = 4$)

Ψ $\Big|_{homogeneous} = 1$

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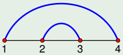
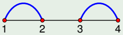
Ψ  $\Big|_{homogeneous} = 1$

Ψ  $\Big|_{homogeneous} = \tau$

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Orbital varieties of order 2

In general, orbital varieties are irreducible components of the intersection of [the closure of] a nilpotent orbit with a Borel subalgebra.
Here we are working with $gl(L)$, and the closure of the nilpotent orbit consists of matrices that square to zero.

$$\mathcal{O} = \{M \text{ upper triangular } L \times L : M^2 = 0\}$$

Fact

These orbital varieties are naturally indexed by link patterns of size L.

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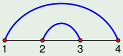
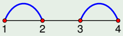
$\mathcal{O}$ 

$$= \left\{ \begin{pmatrix} 0 & m_{13} & m_{14} \\ & m_{23} & m_{24} \\ & & 0 \end{pmatrix} \right\}$$

$\mathcal{O}$ 

$$= \left\{ \begin{pmatrix} m_{12} & m_{13} & m_{14} \\ & 0 & m_{24} \\ & & m_{34} \end{pmatrix} m_{12}m_{24} + m_{13}m_{34} = 0 \right\}$$

Relation to qKZ

Theorem (Di Francesco, PZJ; Knutson, PZJ)

At $q = -1$ i.e. $\tau = 2$, the homogeneous components of the solution of qKZ are the degrees of the corresponding orbital varieties.

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More generally, the full components of rational qKZ correspond to equivariant cohomology classes of these orbital varieties. (wrt conjugation by diagonal matrices and scaling)

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A conjecture that would imply RS

Conjecture (PZJ)

There exists a [Gröbner] [torus-equivariant] degeneration of each orbital variety $\mathcal{O}_\pi$ into a union of complete intersections with only linear and quadratic equations [toric varieties] which are naturally indexed by Fully Packed Loops with connectivity π.

$$\deg \mathcal{O}_\pi = \sum_{\alpha:\ \text{FPL with connectivity } \pi} 2^{n_\alpha}$$

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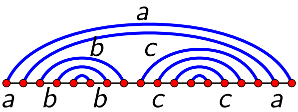
FPLs for three sets of nested arches

Consider a link pattern of the type $\pi =$ 

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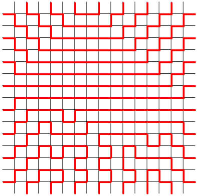
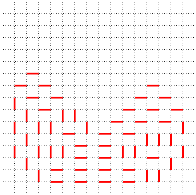
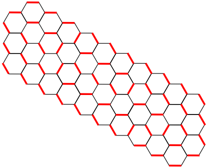
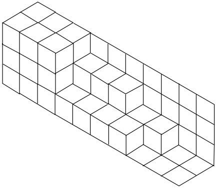
Theorem (Di Francesco, PZJ, Zuber '04)

FPLs with connectivity π are in one-to-one correspondence with plane partitions of size $a \times b \times c$.

Orbital varieties for three sets of nested arches

Fact

The orbital variety corresponding to π is given by $XY = 0$, X $(a+b)\times(b+c)$, Y $(b+c)\times(c+a)$ matrices. [quiver variety]

$$\mathcal{O}_\pi = \left\{ \begin{array}{c} \\ a+b \\ b+c \\ c+a \end{array} \begin{array}{c} \begin{array}{ccc} a+b & b+c & c+a \end{array} \\ \begin{pmatrix} 0 & X & \star \\ & 0 & Y \\ & & 0 \end{pmatrix} \end{array} \qquad XY = 0 \right\}$$

actually, up to some lower dimensional stuff. . .

The degeneration

For each equation defining $\mathcal{O}_\pi$, inside the sum $\sum_j x_{ij} y_{jk}$ keep only the terms of the form $j = i + k - a - 1$ or $j = i + k - a$.
There are either one or two such terms.

When only one term is left, the equation $x_{ij} y_{jk} = 0$ leads to a decomposition into two pieces: $x_{ij} = 0$ or $y_{jk} = 0$. This itself can further simplify some remaining two-term equations, etc.

$\Rightarrow$ at the end of the day we have a number of algebraic varieties given by linear equations of the form $x_{ij} = 0$, $y_{jk} = 0$, and the remaining quadratic equations $x_{ij} y_{jk} + x_{i\,j+1} y_{j+1\,k} = 0$.

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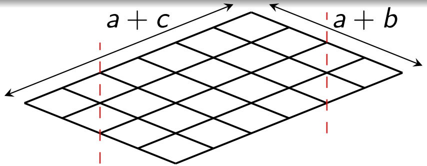

The original set of equations:
$\sum_j x_{ij} y_{jk} = 0$,
$i = 1, \ldots, a+b, \; j = 1, \ldots, a+c$

The cut-off corners correspond to equations that become trivial after degeneration. These trivial (linear) equations cause a "ripple effect" on the remaining quadratic equations:

Each corresponds to the quadratic equations left in the end, whereas and correspond to the two types of linear equations. (two choices for the defect lines)

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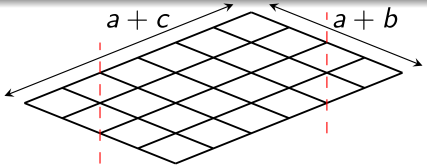

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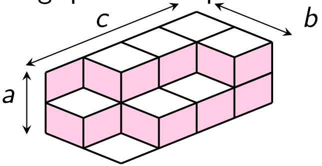
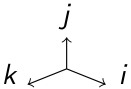
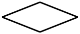




Each  corresponds to the quadratic equations left in the end, whereas  and  correspond to the two types of linear equations. (two choices for the defect lines)