Symplectic Heegaard splittings and generalizations

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A Heegaard splitting of a 3-manifold W is a decomposition of W into two handlebodies, $N_g \cup \bar{N}_g$, where $N_g \cap \bar{N}_g = \partial N_g = \partial \bar{N}_g$, a 2-manifold M_g.

Relate to mapping class group $\mathcal{M}_g$ of ∂N_g as follows: N_g oriented, $\bar{N}_g = e(N_g)$, copy of N_g with inherited orientation. Choose $h \in \mathcal{M}_g$ and choose fixed o.r. map $i : \partial N_g \to \partial N_g$. Let

$$W = N_g \amalg \bar{N}_g / \{x = e \cdot i \cdot h(x),\ \forall x \in \partial N_g\}$$

So each $h \in \mathcal{M}_g$ determines a 3-manifold $W = W(h)$.

Our conventions imply:

$$h = \text{identity} \Longrightarrow W(h) = \#_g S^1 \times S^2 \text{ and } \pi_1(W) = \mathbb{Z}^g.$$

The Johnson-Morita filtration of $\mathcal{M}$

Let $\pi = \pi_1(\partial N_g)$. Define the groups in the lower central series of π, i.e. $\pi^{(1)} = \pi$ and $\pi^{(k)} = [\pi, \pi^{(k-1)}]$. Each $\pi^{(k)}$ a fully invariant subgroup of π.

$\mathcal{M}_g = \pi_0(\mathrm{Diff}^+(\partial N_g))$ acts on π, and action leaves each $\pi^{(k)}$ invariant. So we have an infinite family $\rho^{(k)} : \mathcal{M}_g \to \mathrm{Aut}(\pi/\pi^{(k)})$, which capture more and more information as we increase k. Filtration of $\mathcal{M}_g$.

1985 – started a project with Dennis Johnson.

Question: What can we learn about $W(h)$ from the representation $\rho^{(2)}(\mathcal{M}_g)$, i.e. the symplectic case? Literature review. Reidemeister, 1934. Seifert, 1933. E.Burger 1950. C.T.Wall 1964. Tie it all together. Project started, abandoned in mid 1980's. Manuscript resurfaced 2003. Putman consulted, expressed interest. Recently completed by JB and Putman. Consulted and checked it with Johnson August 2007.

2005 – began work with Tara Brendle and Nathan Broaddas on representation $\rho^{(3)}(\mathcal{M}_g)$. Asked about the new info? Morita had shown image of $\mathcal{M}_{g,1}$ split as semi-direct product of $\wedge^3 H$ (normal subgroup) and $Sp(2g, \mathbb{Z})$ (quotient). So 'new' part separated from 'old part'. Needed to understand old well to study new. e.g. needed normal forms for Sp part of 'gluing map'.

So the 2 projects inter-related.

But Heegaard splittings only one way to represent 3-manifolds. Also surgery, branched covers etc. One expects info to be duplicated. How????

Give some examples of things we can learn about $W(h)$ from $\mathcal{M}_g$:
Example 1: Given h, compute $\pi_1(W(h))$ from $h_\star(\pi_1(\partial N_g))$:

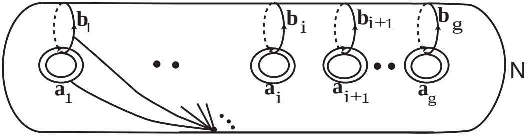
Suppose $\pi_1(\partial N_g,\star) =< a_1,\ldots,a_g,b_1,\ldots,b_g; \prod_{i=1}^{g}[a_i,b_i] >$
Suppose $h_\star(a_i) = A_i(a_1,\ldots,a_g)$, $h_\star(b_i) = B_i(a_1,\ldots,a_g,b_1,\ldots,b_g)$,

Note: $b_1, ..., b_g$ a basis for $\pi_1(N_g)$, $B_1,\ldots,B_g$ a basis for $\pi_1(\bar{N}_g)$.

Van-Kampen Theorem gives us a presentation for $\pi_1(W(h))$:

$$\pi_1(W(h)) =< a_1,\ldots,a_g; B_i(a_1,\ldots,a_g,1,\ldots,1), i=1,\ldots,g >$$

Determined completely by action of h on $\pi_1(N\cap\bar{N})$. But $\pi_1(W)$ an infinite group, so a presentation often reveals very little. So instead look at the nilpotent quotients of $\pi_1(W)$, which can be determined from Johnson filtration.

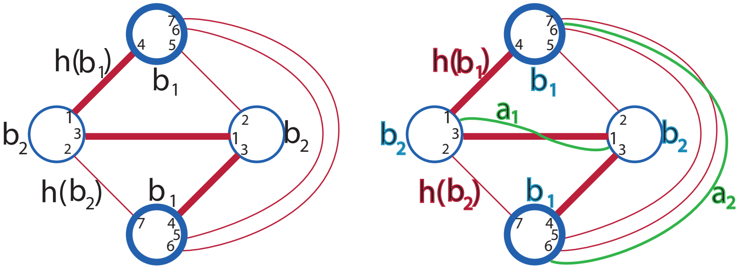


$$\begin{tikzcd} & N \arrow[dr] & \\ N\cap\bar{N} \arrow[ur] \arrow[dr] & & W \\ & \bar{N} \arrow[ur] & \end{tikzcd}$$

All maps are induced by inclusion

Example 2: Mapping class approach gives us an augmented Heegaard diagram: A Heegaard diagram for $W(h)$ is two families of curves, each containing g scc's on $N_g \cap \bar{N}_g$. In our setting the curves are blue curves $b_1, \ldots, b_g$ and red curves $h(b_1), \ldots, h(b_g)$. The Heegaard diagram is $(\partial N_g, \vec{b}, h(\vec{b}))$. It determines $W(h)$ uniquely.

But when we work with h we are studying *augmented* Heegaard diagrams. In general we have $2g$ simple loops a_i, b_i, also $A_i(\vec{a}, \vec{b}), B_i(\vec{a}, \vec{b}), i = 1, \ldots, g$ on ∂N_g, and so a $2g \times 2g$ matrix, each entry a pair of simple loops from the collection. Example:



Heegaard diagram

Augmented Heegaard diagram

Equivalent splittings: Choose $h, h' \in \mathcal{M}_g$.
Define $h \approx h'$ if $\exists$ an orientation-preserving diffeomorphism $F : W(\tilde{h}) \to W(\tilde{h}')$, such that $F(N_g) = N_g,\ F(\bar{N}_g) = \bar{N}_g$.
On $\partial N_g = \partial \bar{N}_g$ have a commutative diagram

$$\begin{array}{ccccc} \partial N_g & \xrightarrow{h} & \partial N_g & \xrightarrow{canonical} & \partial \bar{N}_g \\ \downarrow{\scriptstyle f=F|\partial N_g} & & & & \downarrow{\scriptstyle \bar{f}=F|\partial \bar{N}_g} \\ \partial N_g & \xrightarrow{h'} & \partial N_g & \xrightarrow{canonical} & \partial \bar{N}_g \end{array}$$

Chasing around the diagram, we find:
Let $\mathcal{H}_g = \{f \in \mathcal{M}_g$ such that f extends to $F : N_g \to N_g\}$.
So $h \approx h'$ iff

$$h' \in (\mathcal{H}_g)(h)(\mathcal{H}_g)$$

Stable equivalence of splittings: Assume $h, h' \in \mathcal{M}_g$ inequivalent. Let $s \in \mathcal{M}_1$ be Heegaard gluing map for a genus 1 Heegaard splitting of S^3.
Then $h \approx_s h'$ if there exists u such that

$$(h' \#_u s) \in (\mathcal{H}_{g+u})(h \#_u s)(\mathcal{H}_{g+u})$$

The Reidemeister-Singer Theorem: Any two Heegaard splittings of the same manifold are stably equivalent. Invariants of stable equivalence are topological invariants of $W(h)$. Invariants of Heegaard splittings may or may not be topological invariants.

Example 3: We gave, earlier, a presentation for $\pi_1(W(h))$ that was adapted to a Heegaard splitting:

$$\pi_1(W(h)) =< a_1, \dots, a_g; B_i(a_1, \dots, a_g, 1, \dots, 1), i = 1, \dots, g >$$

G be a group, with ordered generating sets $\mathcal{A} = \{a_1, \dots, a_g\}$ and $\mathcal{A}' = \{a'_1, \dots, a'_g\}$. The generating sets $\mathcal{A}, \mathcal{A}'$ are Nielsen-equivalent if there are bases $\mathcal{X} = \{x_1, \dots, x_g\}$ and $\mathcal{X}' = \{x'_1, \dots, x'_g\}$ for F_g and an epimorphism $\phi : F_n \rightarrow G$ such that $\phi(x_i) = a_i$ and $\phi(x'_i) = a'_i$.

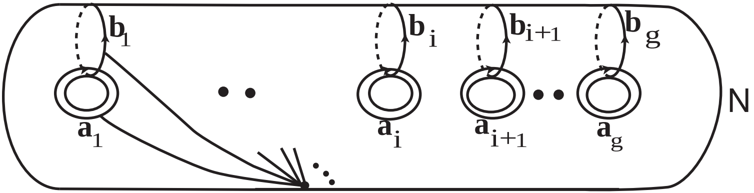
1991: Lustig and Moriah studied Heegaard splittings of certain Seifert fibered spaces. Suspected two splittings were not equivalent. Used Fox derivatives to prove the presentations from their two different Heegaard splittings of $\pi_1(W)$ not Nielsen-equivalent. Argument uniquely adapted to Heegaard splittings of certain SFS's.

A program for studying Heegaard splittings via the Johnson-Morita filtrations of the MCG: Look at the double cosets

$$\rho^{(k)}(\mathcal{H}_g h \mathcal{H}_g) \quad \text{and} \quad \rho^{(k)}(\mathcal{H}_{g+k} h \mathcal{H}_{g+k}).$$

The subgroup $\mathcal{H}_g$ is the handlebody subgroup of $\mathcal{M}_g$. Its structure is unknown in general.

It's the subgroup of all mapping classes on a Heegaard surface ∂N_g which extend to the handlebody N_g. Equivalently, with our conventions, it's the subgroup of Aut $(\pi_1(\partial N_g))$ which preserves the normal closure of $b_1, \ldots, b_g$.



Some questions we might like to answer, using the filtration:

1: Find invariants which characterize minimal (unstabilized) Heegaard splittings at level k and learn how to compute them.

2: Find invariants for stabilized Heegaard splittings at level k, and a constructive procedure for computing them. Note that these will be topological invariants of $W^3(h)$.

3: Determine whether there is a bound on the stabilization index of a Heegaard splitting at level k. Many interesting open questions here. Discuss.

4: Count the number of equivalence classes of minimal (unstabilized) Heegaard splittings at level k

5: Choose unique representatives for unstabilized and stabilized Heehaard splittings at level k.

Overall question: What, if anything, generalizes to double cosets in $\mathcal{M}_g$?

First non-trivial case: $\rho^{(2)} : \mathcal{M}_g \to \mathrm{Aut}(\pi/[\pi.\pi]) = Sp(2g, \mathbb{Z})$.

We call $W(\rho^{(2)}(h))$ a symplectic Heegaard splitting. Describe now recent joint work with Dennis Johnson and Andy Putman [BJP]. Review of contributions of Reidemeister (1935), Seifert (1935), Burger, Wall, JB (1975), Johnson (1985). Some of it not well known. Even when well-known, it's scattered.

$Sp(2g; \mathbb{Z})$ = group of $2g \times 2g$ matrices H such that

$$H = \begin{pmatrix} \mathcal{R} & \mathcal{P} \\ \mathcal{S} & \mathcal{Q} \end{pmatrix} \text{ such that } H^{tr}\mathcal{J}H = \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Equivalently: $H \in \mathrm{Sp}(2g, \mathbb{Z})$ if and only if:

$\mathcal{R}^{\mathrm{tr}}\mathcal{S}, \mathcal{P}^{\mathrm{tr}}\mathcal{Q}, \mathcal{R}\mathcal{P}^{\mathrm{tr}}, \mathcal{S}\mathcal{Q}^{\mathrm{tr}}$ symmetric, and
$\mathcal{R}^{\mathrm{tr}}\mathcal{Q} - \mathcal{S}^{\mathrm{tr}}\mathcal{P} = \mathcal{R}\mathcal{Q}^{\mathrm{tr}} - \mathcal{P}\mathcal{S}^{\mathrm{tr}} = 1$

Want to look at double cosets

$$\rho^{(2)}(\mathcal{H}_g h \mathcal{H}_g) \quad \text{and} \quad \rho^{(2)}(\mathcal{H}_{g+k} h \mathcal{H}_{g+k}).$$

Image of handlebody subgroup of $\mathcal{M}_g$ in $\mathrm{Sp}(g, \mathbb{Z})$ is subgroup of $\mathrm{Sp}(g, \mathbb{Z})$ $\{\rho^{(2)}(\mathcal{H}_g) = \left(\begin{smallmatrix} \mathcal{R} & 0 \\ \mathcal{S} & \mathcal{Q} \end{smallmatrix}\right) \in \mathrm{Sp}(2g, \mathbb{Z})\}$. Has $g \times g$ block of zeros, upper right.

Lemma: The group $\rho^{(2)}(\mathcal{H}_g)$ is the semi-direct product of two subgroups, $\mathbb{S}_g$ and $\mathbb{U}_g$, with $\mathbb{S}_g$ normal, where:

$$\mathbb{S}_g = \{\left(\begin{smallmatrix} I & 0 \\ \mathcal{S} & I \end{smallmatrix}\right), \mathcal{S} \text{ symmetric}\} \text{ and } \mathbb{U}_g = \{\begin{pmatrix} \mathcal{U}^{\mathrm{tr}} & 0 \\ 0 & \mathcal{U}^{-1} \end{pmatrix}, \mathcal{U} \in GL(g, \mathbb{Z})\}.$$

Our double cosets are now modulo semi-direct product of $\mathbb{S}_g \rtimes \mathbb{V}_g$.

The groups $\rho^{(2)}(\mathcal{H}_g)$ (and also $\rho^{(3)}(\mathcal{H}_g)$) are easy to work with.

Open problem: Understand the structure of $\rho^{(k)}(\mathcal{H}_g)$, $k \geq 4$.

Group invariants coming from $\rho^{(2)}(h)$:

(1) Recall that:

$$\pi_1(W(h)) =< a_1, \dots, a_g; B_i(a_1, \dots, a_g, 1, \dots, 1), i = 1, \dots, g >.$$

This gives, immediately, a related presentation for $H_1(W(h), \mathbb{Z})$.

It follows that if $\rho^{(2)}(h) = H(h) = \begin{pmatrix} \mathcal{R} & \mathcal{P} \\ \mathcal{S} & \mathcal{Q} \end{pmatrix}$, then the $g \times g$ submatrix $\mathcal{P}$ is a relation matrix for $H_1(W, \mathbb{Z})$. Since $H_1(W, \mathbb{Z})$ is a f.g. abelian group, it's a direct sum of r infinite cyclic groups and t finite cyclic groups of orders $\tau_1, \dots, \tau_t$, where each τ_i divides τ_{i+1}. Topological invariants of W are:

(a) r = torsion-free rank of $H_1(W)$
(b) torsion coefficients $\tau_1, \dots, \tau_t$. of $H_1(W)$
(c) number of homologically trivial summands in presentation.

Fundamental theorem of f.g. abelian groups says any two presentations equivalent under a change in basis $\Longrightarrow$ left and right multiplication of H by elements in $\mathbb{V}$ brings $\mathcal{P}$ to diagonal form $\mathrm{diag}(0, \dots, 0, \tau_1, \dots, \tau_t, 1, \dots, 1)$.

Having the diagonal form for $\mathcal{P}$, can show: the double coset of $H(h)$ in $\mathrm{Sp}(2g,\mathbb{Z})$ contains a matrix of the form:

$$\mathbb{H}'(h) = \left(\begin{array}{ccc|ccc} 1_r & 0 & 0 & 0_r & 0 & 0 \\ 0 & \mathcal{R}'_t & 0 & 0 & \mathcal{T}'_t & 0 \\ 0 & 0 & 0_{g-r-t} & 0 & 0 & 1_{g-r-t} \\ \hline 0_r & 0 & 0 & I_r & 0 & 0 \\ 0 & \mathcal{S}'_t & 0 & 0 & \mathcal{Q}'_t & 0 \\ 0 & 0 & -I_{g-r-t} & 0 & 0 & 0_{g-r-t} \end{array}\right)$$

where $\mathcal{T}' = diag(\tau_t, \ldots, \tau_1)$.

The interest is all in the submatrix $\mathbb{T}(h)) = \begin{pmatrix} \mathcal{R}' & \mathcal{T}' \\ \mathcal{S}' & \mathcal{Q}' \end{pmatrix}$ associated to the torsion subgroup of $H_1(W)$, where $\mathcal{T}' = diag(\tau_t, \ldots, \tau_1)$.

A linking form on a finite abelian group $\mathbb{T}$ is a symmetric bilinear form on $\mathbb{T}$, with values in the rationals mod 1. In the case of the torsion subgroup $\mathbb{T}(h)$ of $H_1(W)$, the matrix $\mathcal{Q}'(\mathcal{T}')^{-1} = (\lambda_{ij})$ is a linking form on the torsion subgroup $\mathbb{T}(h)$ of $H_1(W(h))$. Each $\lambda_{ij} \in \mathbb{Q}$ (mod 1). Determined by $(\mathcal{T}'_t, \mathcal{Q}'_t)$. So the pair $(\mathcal{T}'_t, \mathcal{Q}'_t)$ contains info on torsion and linking.

Invariants of the linked abelian group Two canonical ways to decompose a finite abelian group. Pass to the second. Torsion coefficients are $\tau_1, \ldots, \tau_t$, and each τ_i is a product of powers of primes:

$$\tau_i = p_1^{e_{i,1}} p_2^{e_{i,2}} \cdots p_k^{e_{i,k}}, \quad 0 \leqslant e_{1,d} \leqslant e_{2,d} \leqslant \cdots \leqslant e_{t,d}, \text{ for each } 1 \leq d \leq k.$$

Two cases, according as all p_i are odd (easier case) or (harder case) there is 2-torsion. Seifert studied odd case.

Theorem (Seifert, 1935)

Every linking form on T splits as a direct sum of linkings associated to the p-primary summands of T, and two linking forms are equivalent if and only if the linkings on the summands are equivalent.

Seifert studied linking matrix $\lambda(g_i, g_j)$ belonging to the subgroup $\mathcal{T}(p) \subset \mathcal{T}$ of cyclic summands whose order is a power of a fixed prime $p = p_d$. The linking matrix divides into blocks whose size is determined by the number of times t_i that a given power, say p_i^{ε}, is repeated. Among these, the blocks that interest us are the square blocks whose diagonals are along the main diagonal of the linking matrix.

There will be r such blocks of dimension $t_1, \ldots, t_r$ if r distinct powers p^{ε_i} occur in the subgroups of $\mathcal{T}$ that are cyclic with order a power of p:

$$\left\|\lambda(g_i, g_j)\right\| = \begin{pmatrix} \frac{\mathcal{A}_1}{p^{\varepsilon_1}} & * & \cdots & * \\ * & \frac{\mathcal{A}_2}{p^{\varepsilon_2}} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & \cdots & \cdots & \frac{\mathcal{A}_r}{p^{\varepsilon_r}} \end{pmatrix} \quad (1)$$

The stars relate to linking numbers that we shall not consider further.

Theorem (Seifert)

If p is odd, two linkings of $\mathcal{T}(p)$ are equivalent if and only if the corresponding box determinants $|\mathcal{A}_1|, |\mathcal{A}_2|, \ldots, |\mathcal{A}_r|$ have the same quadratic residue characters $\mod p$.

Seifert's invariants readily computable. But he could not handle the case when there is 2-torsion.

If $H_1(W(h))$ has 2-torsion, the linking splits as before into a direct sum of linkings on the p-primary components $\mathcal{T}(p_j)$ but the linking type of $\lambda|_{\mathcal{T}(p_j)}$ *is no longer an invariant*. Need to work much harder.

Burger showed how to decompose linked abelian 2-group into orthogonal direct sum of 3 basic linking forms:

- The *unary forms*. Forms on $\mathbb{Z}_{2^j}$ for $j \geq 1$ whose matrices are $\left(\frac{a}{2^j}\right)$ for odd integers a.
- The two *binary forms*. Forms on $(\mathbb{Z}_{2^j})^2$ for $j \geq 1$ whose matrices are either $\frac{1}{2^j}\mathcal{C}$ or $\frac{1}{2^j}\mathcal{D}$, where

$$\mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

Burger's work improved by Wall, also Fox. See B-J-P for simplified picture and methods of computation of complete set of invariants.

Invariants of unstabilized Heegaard splittings

'Unstabilized', in the setting of symplectic Heegaard splittings, means that the diagonal matrix $\mathcal{P} = \mathrm{diag}(0, \ldots, 0, \tau_t, \ldots, \tau_1)$ has no unit entries. We find invariants of minimal symplectic HS $\rho^{(2)}(W(h))$ the presentation of the pair $(\mathcal{T}', \mathcal{Q}')$, a finite abelian group with a linking.

Theorem

a) *Assume h_1, h_2 define minimal symplectic Heegaard splittings. Assume splittings are stably isomorphic.*

1. *If $16 \nmid \tau_1$, then $h_1 \approx h_2 \iff \det(\mathcal{Q}_1') = \det(\mathcal{Q}_2') \mod (\tau_1)$.*
2. *If $16 | \tau_1$, then $h_1 \approx h_2 \iff \det(\mathcal{Q}_1') = \det(\mathcal{Q}_2') \mod 2\tau_1$.*

b) *The number of equivalence classes of minimal SHS is finite. Its order depends on number theoretic properties of τ_1.*

c) *(Unfortunately), any two non-minimal symplectic Heegaard splittings of the same 3-manifold are equivalent. That is, index of stabilization is 1.*

Open problem: Better understanding of inequivalent minimal Heegaard splittings. Limited information in $\rho^{(2)}$ level representation of $\mathcal{M}_g$.

Now study $\rho^{(3)}(\mathcal{M}_g)$, asking the same questions. Joint work with Tara Brendle and Nathan Broaddus.

Goal is same as in case of $\rho^{(2)}$: What can you learn about the problems described earlier? Want two kinds of information: about minimal Heegaard splittings and fully stabilized Heegaard splittings, and index of stabilization.

Technical issue: $\psi : \mathcal{M}_{g,1} \to \mathcal{M}_g \supset H_g$. Define $\mathcal{H}_{g,1} = \psi^{-1}(\mathcal{H}_g)$. Define $S_g = \partial N_g$.

Lemma

The handlebody subgroup $\mathcal{H}_{g,1}$ of the mapping class group $\mathcal{M}_{g,1} \subset \mathrm{Aut}(\pi_1 S_{g,1}))$ is the subgroup which preserves $\mathfrak{b}$ the kernel of the homomorphism $\pi_1(S_{g,1}) \to \pi_1(N_g)$ induced by the inclusion map.

Proof: One direction clear. Assume $f \in \mathcal{M}_{g,1}$ preserves $\mathfrak{b}$. Then f sends each b_i to a loop that can be represented by a simple closed curve which is trivial in $\pi_1(N_g)$. Loop theorem shows curves that bound disks in N_g can be made disjoint. Matching these disks to the ones bounded by each b_i, construct a homeomorphism of N_g restricting to f on ∂N_g.

Important fact: Morita proved $\rho^{(3)}(\mathcal{M}_g)$ can be identified with a subgroup of $\left(\frac{1}{2} \wedge^3 H\right) \rtimes \mathrm{Sp}(H)$. Use Morita's version of $\rho^{(3)}$ to produce our Heegaard invariants.

Need to know $\rho^{(3)}(\mathcal{H}_{g,1})$.

Corollary (Image of the handlebody subgroup under $\rho^{(3)}$)

Let $R \in \mathrm{Sp}(2g, \mathbb{Z})$. Let r be any element of $\frac{1}{2} \wedge^3 H$ with

$$r = \sum_{1 \leq i<j<k \leq 2g} r_{ijk} x_i \wedge x_j \wedge x_k.$$

Then $(r, R) \in \rho^{(3)}(\mathcal{H}_{g,1}) \iff$ the following 3 conditions hold:

1. $R \in \rho^{(2)}(\mathcal{H}_g)$
2. *r contains no terms of the form $a_i \wedge a_j \wedge a_k$.*
3. *r_{ijk} satisfies certain conditions (you don't want to see them now).*

In particular if R has the required block form and we set $r_R = \sum_{1 \leq i<j<k \leq 2g} \frac{1}{2} E_{ijk} x_i \wedge x_j \wedge x_k$ then $(r_R, R) \in \rho^{(3)}(\mathcal{H}_{g,1})$.

As for $\rho^{(2)}$, the description of double cosets depends on $H_1(W(h))$. For $\rho^{(2)}$, the interesting case was when $H_1(W(f))$ finite.

For $\rho^{(3)}$, most accessible case is $\rho^{(3)}(h) \in$ Torelli subgroup of $\mathcal{M}_{g,1}$, i.e. (by our conventions) $H_1(W(h)) = \mathbb{Z}^g$.

Theorem

Assume $h \in \mathcal{I}_{g,1}$. Johnson homomorphism $\tau : \mathcal{I}_{g,1} \to \wedge^3 H$. Let $\wedge^3 H_a =$ subgroup of $\wedge^3 H$ generated by all $a_i \wedge a_j \wedge a_k$. Let $j : \wedge^3 H \to \wedge^3 H_a$ be projection map. Then a complete invariant of stable double costs of $\rho^{(3)}(h)$ is the $GL(g, \mathbb{Z})$-orbit of $j \circ \tau(h) \in \wedge^3 H$ under changes in basis for H_a.

Actually, stronger result holds: It's an invariant of the stable double coset, not just double coset.

Conclusion: Have a new topological invariant of 3-manifolds which have same homology as $\#_g(S^2 \times S^1)$, and it lives in $\wedge^3 H$.

Sketch proof of theorem: We know a complete invariant of the double coset of $\rho^{(3)}(h)$ is

$$Y_h = \left\{ w \in \frac{1}{2} \wedge^3 H \middle| (w, I) \in \mathcal{H}^{(3)} \rho^{(3)}(h) \mathcal{H}^{(3)} \right\}.$$

Putting it another way: $Y_h =$ set of all $w \in \frac{1}{2} \wedge^3 H$ such that there are $(v, V), (u, U) \in \rho^{(3)}(\mathcal{H}_{g,1})$ such that $(v, V)(\tau(h), I)(u, U) = (w, I)$.

Using the rule for the semi-direct product, and multiplying things out, we see that $VU = I$, or $U = V^{-1}$., which implies that $(v, V)(u, U) = (p, I)$ for some $p \in \rho^{(3)}(\mathcal{I}_{g,1} \cap \mathcal{H}_{g,1})$. This shows that:

$$Y_h = \left\{ w \in \frac{1}{2} \wedge^3 H \middle| w = V\tau(h) + p \text{ for some } V \in Sp \text{ and } p \in \rho^{(3)}(\mathcal{H}_{g,1}) \right\}$$

By assumption $h \in \mathcal{I}$, so $\tau(h))$ and p are in $\wedge^3 H$. No fractional coefficients appear, so we can replace $\frac{1}{2} \wedge^3 H$ by $\wedge^3 H$ in Y_h.

Recall $H_a =$ subgp of H generated by $a_1, \ldots, a_g$, and $\wedge^3 H_a \subset \wedge^3 H$ is subgp generated by all $a_i \wedge a_j \wedge a_k$, and $j : \wedge^3 H \to \wedge^3 H_a$ is projection map. Then $\mathcal{I}_{g,1} \cap \mathcal{H}_{h,1} = \ker j$. We proved

$$Y_h = j^{-1}(\{w \in \wedge^3 H_a \big| w = j(V\tau(h)) \text{ for some } V \in \rho^{(2)}(\mathcal{H}_{g,1}\}) \qquad (2)$$

Remember V has a block of zeros in upper right corner. The value of $j(V\tau(h))$ depends only on $j(\tau(h))$ and the upper left $g \times g$ block of $V \in$ Sp. So upper left $g \times g$ block in V ranges over all of $\mathrm{GL}(g, \mathbb{Z})$. The double coset completely determined by the orbit of $j(\tau(h))$ under $\mathrm{GL}(H_a)$.

These orbits, and the third nilpotent quotient $\pi_1(W(h))/(\pi_1^{(3)}(W(h))$ are the $\rho^{(3)}$-invariants that we know now, when $h \in \mathcal{I}_{g,1}$.

Is the $\wedge^3 H$ part new? No, it turns out that it was also discovered by Cochran, Gerges and Orr, *Dehn surgery equivalence relations in 3-manifolds*, Math Proc. Camb. Phil, Soc 2001. Beautiful paper, uses different approach.

Orbits of $x \in \wedge^3\mathbb{Z}^g$ under $\mathrm{GL}(g,\mathbb{Z})$ are a complicated set. Deciding if two elements of $\wedge^3\mathbb{Z}^g$ are in the same $\mathrm{GL}(g,\mathbb{Z})$ orbit is an interesting problem in its own right.

Computable invariant of the orbit of $z \in \wedge^3\mathbb{Z}^g$ is the *GCD* of the coefficients of z. It's constant on the orbit. An integer. Is it a known invariant?

Secondly, let V be a real g-dimensional vector space. Let $W \subset V$ be a subspace. We will say that an element $w \in \wedge^3 V$ is *supported* by W if there are $w_i \in \wedge^3 W$ such that $w = \sum_{i=1}^n w_i$. The minimum of the set $\{\dim(W) | W \text{ supports } z \otimes \mathbb{R}\}$ is an invariant of the orbit of $z \otimes \mathbb{R}$ under the action of $\mathrm{GL}(g,\mathbb{R})$ and hence another invariant of the orbit of z under $\mathrm{GL}(g,\mathbb{Z})$.

Wild guess: the Thurston norm is in here someplace.

Remark: Can do the same thing if h not in Torelli, but Y_h is known less precisely.

Remark: The level 3 invariants just described are all computable.

Some general remarks.

At every level we have the corresponding nilpotent quotient of $\pi_1(W(h))$. Have these been studied? Don't know.

Three-manifold invariants: Casson's invariant enters at level 4. Beyond level 4, nothing known.

Rohlin invariant – Johnson's 'spin mapping class group'.

Connection with Vassiliev invariants needs to be pinned down. Expect more than one topological invariant at level k, from experience with Vassiliev invariants of knots. Have seen nothing like this so far.

Invariants of Heegaard splittings are needed. Are there lifts of the linking form? Associated invariants at level k? Subtle question. Stabilization issues.

New work of Hass-Thompson-Thurston on index of stabilization. Should be able to detect their examples algebraically. Many many open questions.

Another open problem: filtration $\rho^{(k)}(\mathcal{H}_g)$ of $\mathcal{H}_g$.

For $k = 2$:

$$\mathbb{S}_g = \{\left(\begin{smallmatrix} I & 0 \\ \mathcal{S} & I \end{smallmatrix}\right), \mathcal{S} \text{ symmetric}\} \text{ and } \mathbb{U}_g = \{\begin{pmatrix} \mathcal{U}^{\mathrm{tr}} & 0 \\ 0 & \mathcal{U}^{-1} \end{pmatrix}, \mathcal{U} \in GL(g, \mathbb{Z})\}.$$

$$\rho^{(2)} : \mathcal{M}_g \to Sp(2g, \mathbb{Z}), \quad \text{and} \quad \rho^{(2)}(\mathcal{H}_g) = \mathbb{S}_g \rtimes \mathbb{U}_g$$

For $k = 3$:

$$\wedge^3 H_a = \{a_i \wedge a_j \wedge a_k \in \wedge^3 H\}$$

$$\rho^{(3)} : \mathcal{M}_g \to \left(\frac{1}{2} \wedge^3 H\right) \rtimes \mathrm{Sp}(H), \quad \mathcal{H}_g \to \wedge^3 H_a \rtimes \rho^{(2)}(\mathcal{H}_g)$$

$k = 4$??

Wajnryb: Presentation for $\mathcal{H}_g$. His work needs attention.