A Topological Construction of Crossed Homomorphisms Extending the Higher Johnson Homomorphisms

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Outline

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The Basic Objects of Study

- Let $\Sigma = \Sigma_{g,1}$, with $g \geq 3$ and basepoint $* \in \partial\Sigma$.

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- Let $Mod_{g,1}$ be the mapping class group relative to boundary.

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- The *Johnson Filtration*:

$$Mod_{g,1} = \mathcal{I}_{g,1}(1) \supset \mathcal{I}_{g,1}(2) \supset \cdots \supset \mathcal{I}_{g,1}(k) \supset \cdots$$

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Definition (Johnson)

The *kth Johnson homomorphism* τ_k is $\rho_{k+1}|_{\mathcal{I}_{g,1}(k)}$, with its range lifted to $\operatorname{Hom}(H, \mathcal{L}_{k+1})$:

$$\tau_k \colon \mathcal{I}_{g,1}(k) \to \operatorname{Hom}(H, \mathcal{L}_{k+1})$$

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- The Hochschild-Serre spectral sequence for

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has a differential:

$$d^2\colon\ H_3(\Gamma_k) \to H_1(\Gamma_k, H_1(\mathcal{L}_{k+1})) \cong \mathrm{Hom}(H, \mathcal{L}_{k+1})$$

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- Compare to $\ker \tau_k = \mathcal{I}_{g,1}(k+1)$.

Other Work Extending Johnson Homomorphisms

	Morita 1993	Morita 1996	Hain 1997	Perron 2004	Kawazumi 2005
Crossed homom. to abelian group	$k = 2$		$k = 2$	$k = 2$	$k = 2$
Crossed homom. to nilpotent group		$k = 3$	$\forall k \geq 3$	$k = 3$	
Other extension				$k = 3$	$\forall k \geq 3$

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Results on Morita's Homomorphisms

Theorem (M. Day 2007)

For each k, there is a finite-dimensional $\mathbb{R}$–vector space V_k with a $Mod_{g,1}$ action, such that $H_3(\Gamma_k) \hookrightarrow V_k$ equivariantly and there is a crossed homomorphism $\epsilon_k\colon Mod_{g,1} \to V_k$ extending $\tilde{\tau}_k$. This defines a nontrivial $[\epsilon_k] \in H^1(Mod_{g,1}; V_k)$.

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Corollary

$$Mod_{g,1}/\mathcal{I}_{g,1}(2k-1) \hookrightarrow (Mod_{g,1}/\mathcal{I}_{g,1}(k)) \ltimes V_k$$
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For each k, there is a finite-dimensional $\mathbb{R}$–vector space V_k' with a $Mod_{g,1}$ action, such that $\mathrm{Hom}(H, \mathcal{L}_{k+1}) \hookrightarrow V_k'$ equivariantly and there is a crossed homomorphism $\epsilon_k' \colon Mod_{g,1} \to V_k'$ extending τ_k. This defines a nontrivial $[\epsilon_k'] \in H^1(Mod_{g,1}; V_k')$.

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For each k, there is a finite-dimensional $\mathbb{R}$–vector space V'_k with a $Mod_{g,1}$ action, such that $\mathrm{Hom}(H, \mathcal{L}_{k+1}) \hookrightarrow V'_k$ equivariantly and there is a crossed homomorphism $\epsilon'_k \colon Mod_{g,1} \to V'_k$ extending τ_k. This defines a nontrivial $[\epsilon'_k] \in H^1(Mod_{g,1}; V'_k)$.

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- In fact, without loss of generality, we can take F_ϕ to be a homotopy relative to $\partial\Sigma$.

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The *topological version of the kth Morita homomorphism* $\bar{\epsilon}_k$ is:

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The maps $\tilde{\tau}_k$ *and* $\overline{\epsilon}_k$ *are identified by the canonical isomorphism* $H_3(\Gamma_k) \cong H_3(X_k)$.

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Proof sketch:

- Show that $\bar{\epsilon}_k([\phi])$ is represented by the difference of $(F_\phi)_*(C_\Sigma \times [0,1])$ and a correction factor $D_{\phi^{-1}}$ (C_Σ is a fundamental class rel. to ∂).

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- Show that $(F_\phi)_*(C_\Sigma \times [0,1])$ is sent to a boundary by a chain map inducing the canonical $H_3(\Gamma_k) \cong H_3(X_k)$.
- Show that $D_{\phi^{-1}}$ is sent to a chain representing $\tilde{\tau}_k([\phi])$.

$\square$

$\tilde{\tau}_k$ Definition $\bar{\epsilon}_k$ Definition

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2. But these F_ϕ do not define cycles.
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   - We want an equivariant map from $C_3(X_k)$ to a nicer $Mod_{g,1}$–module.

Outline

General terms

- We will be using Mal'cev completions:

Theorem (Mal'cev)

For each finitely-generated torsion-free nilpotent group Γ, there is a unique contractible nilpotent Lie group G with $\Gamma \hookrightarrow G$ as a lattice. This G is the Mal'cev completion of Γ.
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- Example: The 3–dimensional Heisenberg group.
- Let G_k be the Mal'cev completion of Γ_k. Fix $X_k = G_k/\Gamma_k$. So $Mod_{g,1}$ acts on X_k, fixing the first problem.

More About Nilpotent Homogeneous Spaces

Theorem (Nomizu)

Let X be a compact manifold that is a homogeneous space of a connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$. Then the left-propagation map $L\colon C^*(\mathfrak{g}) \to C^*(X;\mathbb{R})$ induces an isomorphism $H^*(\mathfrak{g}) \to H^*(X;\mathbb{R})$.

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- For $C \in C_m(X;\mathbb{R})$, there is a unique $v(C) \in C_m(\mathfrak{g})$ such that for each $\alpha \in C^m(\mathfrak{g})$:

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- The chain map $v\colon C_*(X;\mathbb{R}) \to C_*(\mathfrak{g})$ induces $H_*(X;\mathbb{R}) \cong H_*(\mathfrak{g})$.

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Nilmanifold "Approximations" to Σ

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- Set $V_k = C_3(\mathfrak{g}_k)/B_3(\mathfrak{g}_k)$.
- By Igusa–Orr (2001), $H_3(\Gamma_k)$ is torsion-free. So we have a $Mod_{g,1}$–equivariant embedding:

$$H_3(X_k) \hookrightarrow H_3(X_k;\mathbb{R}) \cong H_3(\mathfrak{g}_k) \hookrightarrow V_k$$

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The *extended kth Morita map* is:

$$\begin{aligned} \epsilon_k \colon\ Mod_{g,1} &\to V_k \\ [\phi] &\mapsto \Big[v\Big((F_\phi)_*(C_\Sigma \times [0,1])\Big)\Big] \end{aligned}$$

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- ϵ_k extends $\bar{\epsilon}_k$, and $[\epsilon_k] \neq 0$.

Theorem Statement

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- Let $\mathfrak{l}_{k+1}$ be defined by:

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- $H_1(\mathfrak{g}_k; H_1(\mathfrak{l}_{k+1})) \hookrightarrow V'_k$ and there is a $Mod_{g,1}$–equivariant $\tilde{d}^2\colon\ V_k \to V'_k$ extending d^2.

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Define $\epsilon_k' = \tilde{d}^2 \circ \epsilon_k$.

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\begin{tikzcd}
\mathcal{I}_{g,1}(k) \arrow[rr, "\tau_k"] \arrow[dr, "\tilde{\tau}_k"] \arrow[ddr, "\epsilon_k|_{\mathcal{I}_{g,1}(k)}"'] & & \mathrm{Hom}(H, \mathcal{L}_{k+1}) \arrow[d] \\
& H_3(\Gamma_k) \arrow[r, "d^2"] \arrow[d, hook] & H_1(\Gamma_k; H_1(\mathcal{L}_{k+1})) \arrow[d, hook] \\
& H_3(\mathfrak{g}_k) \arrow[r, "d^2"] & H_1(\mathfrak{g}_k; H_1(\mathfrak{l}_{k+1}))
\end{tikzcd}
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- Example: There is a connection between extended flux on $\mathrm{Symp}(\Sigma_{g,*}, \omega)$ and τ_2.
- Remark: There is a related topological construction for τ_k on $\mathrm{Aut}\, F_n$ (*in progress*).
- Question: What is the "best possible" range for a crossed homomorphism extending τ_k?