Dimensions of Torelli groups

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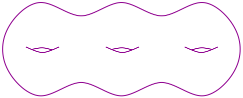
Torelli groups

S_g = surface of genus g



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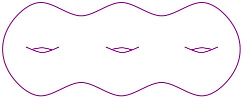

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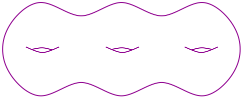

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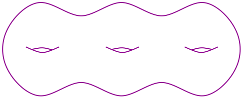

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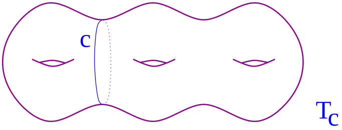
Definition of the Torelli group $\mathcal{I}(S_g)$:

$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z}) \to 1$$

Elements of the Torelli group

Dehn twists about separating curves



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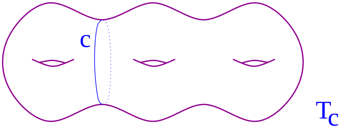
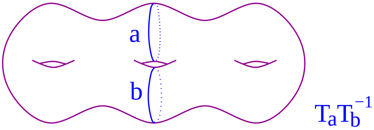


Bounding pair maps



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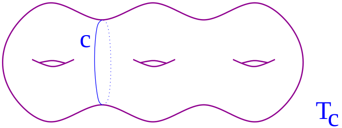

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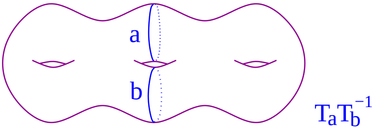

Theorem (Birman '71 + Powell '78)

These elements generate $\mathcal{I}(S_g)$.

Finiteness properties

Finite generation

Finite presentability

Finite generation of homology

Cohomological dimension

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Harer 1986, Culler–Vogtmann 1986 + Mess 1990, Ivanov 1984:

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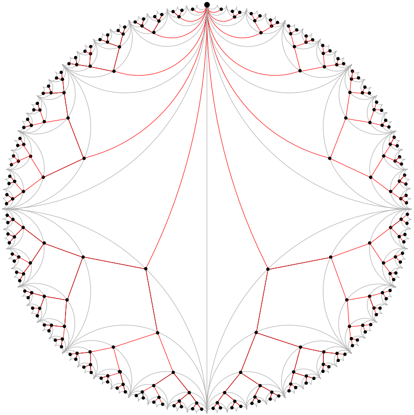
The quotient tree for $\mathcal{I}(S_2)$ is infinitely many copies of



glued along their distinguished vertices.

Dimensions of mapping class groups

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$$\begin{aligned} \mathrm{vcd}(\mathrm{Mod}(S_g)) &= 4g - 5 \\ \mathrm{cd}(\mathcal{I}(S_g)) &= 3g - 5 \end{aligned}$$

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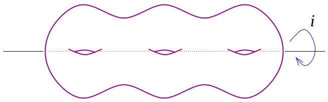

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\mathrm{cd}(\mathcal{K}(S_g)) &= 2g-3 \\
\mathrm{cd}(\mathcal{SI}(S_g)) &= g-1 \quad \text{(Brendle–M)}
\end{aligned}$$

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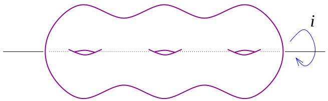

Proof that

$$\mathrm{cd}(\mathcal{I}(S_g)) \leq 3g - 5$$

A bound on cohomological dimension due to Quillen

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$$\mathrm{cd}(G) \leq \sup\{\mathrm{cd}(\mathrm{Stab}(\sigma)) + \dim(\sigma)\}$$

where the supremum is over cells σ of X.

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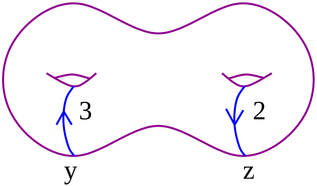
Example: $x = 3y + 2z$



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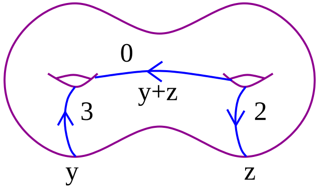

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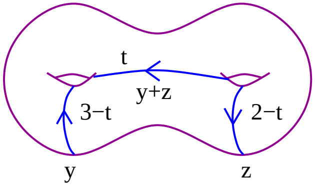

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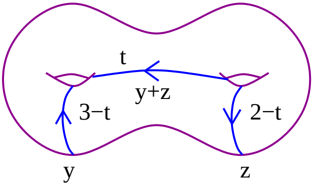

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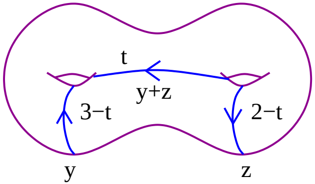
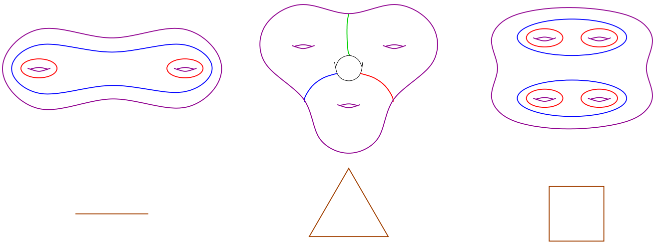


Nonnegativity $\rightsquigarrow 0 \leq t \leq 2$. Resulting cell:

Examples of cells



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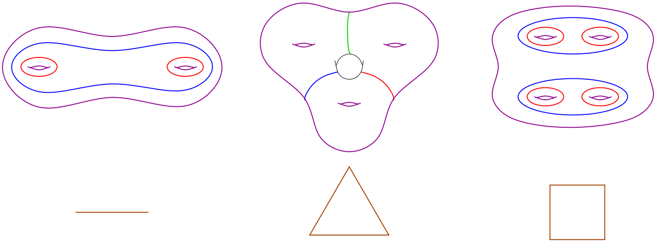

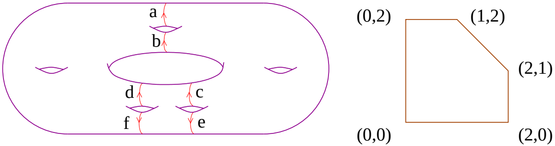

$$x = [d] + 2[e] + [f]$$

The Complex of Minimizing Cycles
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$$\mathrm{Cell}(M) = \{\, c \in \mathbb{R}^{\mathscr{S}} \;:\; \begin{array}{l} c \text{ nonnegative}, \\ c \text{ supported in } M, \\ c \text{ represents } x \,\} \end{array}$$

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Fact: $\mathrm{Cell}(M)$ is a polytope.

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Theorem (BBM)

$\mathcal{B}(S_g)$ is contractible.

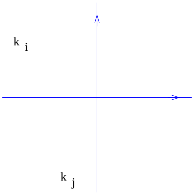
The Complex of Minimizing Cycles

New proof of contractibility

Surgery on 1-cycles

Let c be a nonsimple 1-cycle representing x.

$$c = \sum k_i c_i$$



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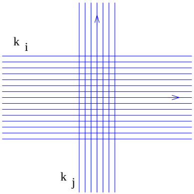

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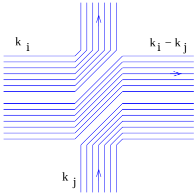

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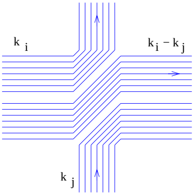

Fact: The result of surgery is a 1-cycle.

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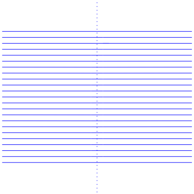

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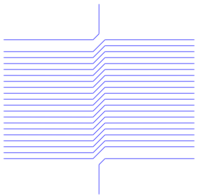

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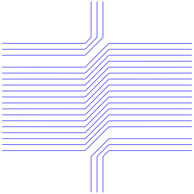

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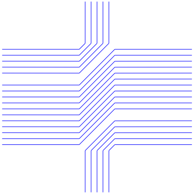

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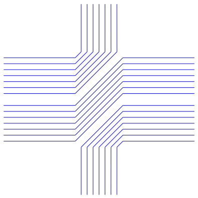

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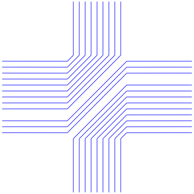

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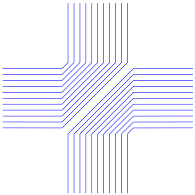

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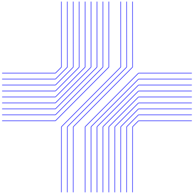

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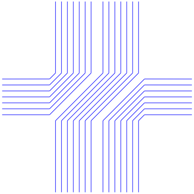

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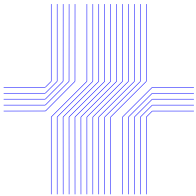

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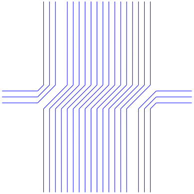

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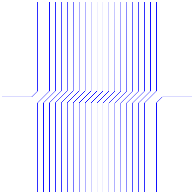

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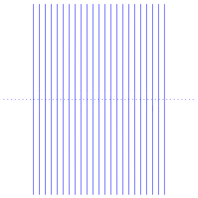

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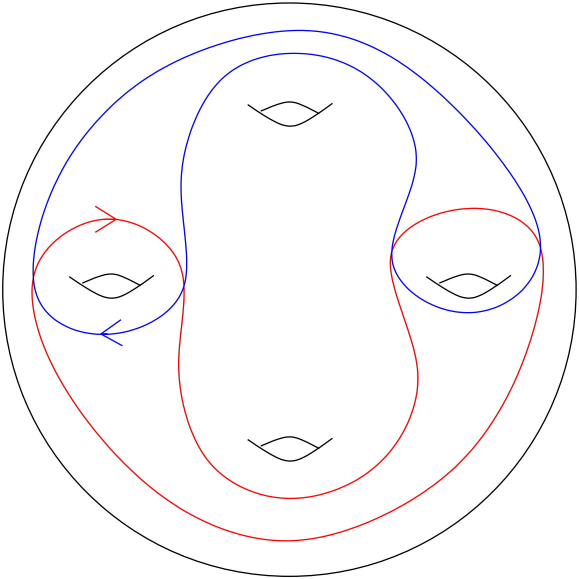

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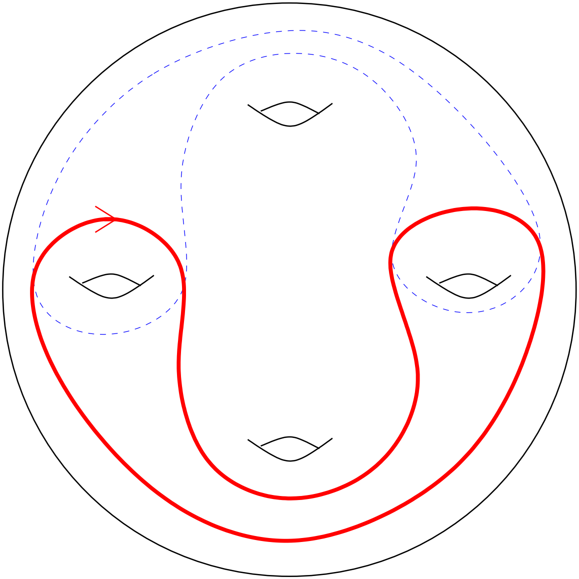

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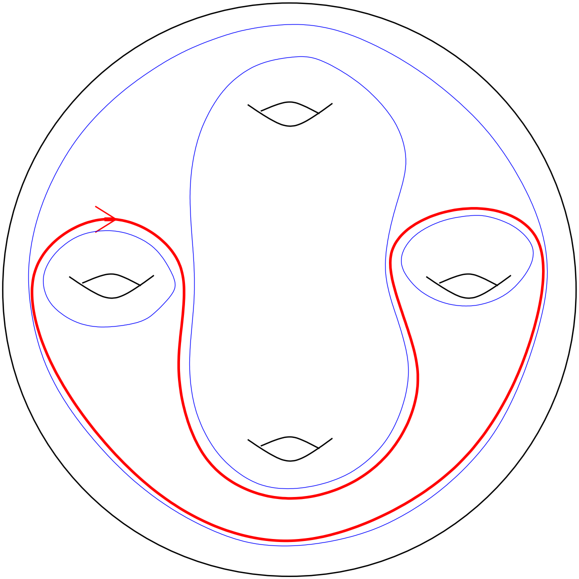

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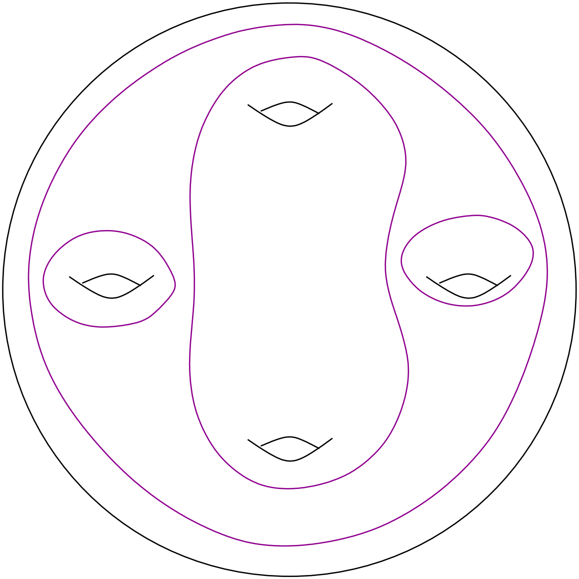

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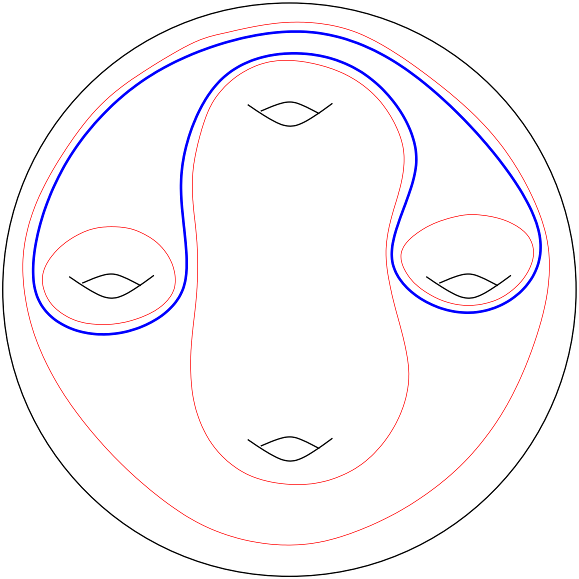

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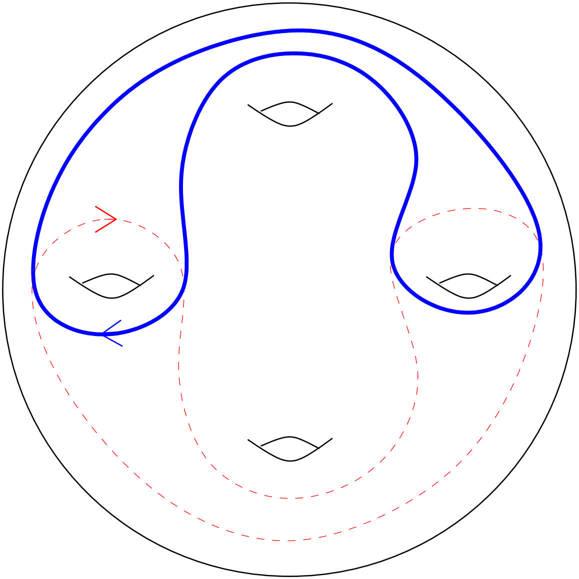

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Recall Quillen condition:

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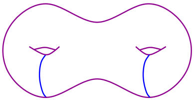

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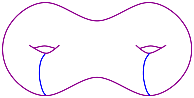
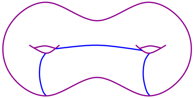


and stabilizers of edges are trivial (0-dimensional).



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Therefore, to prove that (H_1 of) $\mathcal{I}(S_2)$ is infinitely generated, we just need to show that H_1 of some vertex stabilizer is infinitely generated.

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Original proof of contractibility

Idea: Build analogy with Teichmüller space $\mathcal{T}(S)$.

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The proofs of the analogous theorems are incongruous.