Automorphism Groups of Nilpotent Quotients of Fundamental Groups of Surface

and Homology Cobordisms of 3-Manifolds

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1. Ultimate goal : MCG vs group of homology cylinders over $\Sigma_{g,1}$
2. Symplectic automorphism groups of free nilpotent groups
3. Rational forms of the symplectic automorphism groups
4. Group version of the trace maps
5. A representation of $\mathcal{H}_{g,1}$
6. Cohomology classes of $\mathcal{H}_{g,1}$

1. Ultimate goal and strategy, $\mathcal{M}_{g,1}$ vs $\mathcal{H}_{g,1}$

$\mathbf{\Theta}^3$:Group of homology cobordism classes of homology 3-spheres

$$\rho : \mathbf{\Theta}^3 \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}_2 \quad \text{(Rohlin invariant)}$$

For some time, there was a weak conjecture that the above is isomorphic

but, Furuta, Fintushel-Stern proved: $\mathbf{\Theta}^3$ has infinite rank, using gauge theory

Brieskorn spheres $\Sigma(2, 3, 6k - 1)$ $(k = 1, 2, \cdots)$ are linearly independent

$$0 \longrightarrow \mathrm{Ker}\rho \longrightarrow \mathbf{\Theta}^3 \xrightarrow{\rho} \mathbb{Z}_2 \longrightarrow 0$$

split ? $\Leftrightarrow$ triangulability of topological manifolds

Matumoto, Galewski and Stern

known homomorphisms

$$\mathbf{\Theta}^3 \to \mathbb{Q}$$

those defined by Frϕyshov and Ozsváth-Szabó (Heegaard Floer homology)

but it seems that they are conjectured to be equal...

candidate: Neumann-Siebenmann, Fukumoto-Furuta-Ue, Saveliev

$$\nu := \sum_{i=0}^{7} (-1)^{\frac{i(i+1)}{2}} \text{rank } HF^i \text{ (instanton Floer homology)}$$

recall:

$$\sum_{i=0}^{7} (-1)^i \text{rank } HF^i = 2\lambda \text{ Casson invariant}$$

Taubes

$\Rightarrow$ want to define homomorphisms

$$\mathbf{\Theta}^3 \to \mathbb{Q}$$

extremely difficult problem, just "an atempt" for the moment...

Garoufalidis-Levine introduced (based on works of Goussarrov and Habiro)

$\mathcal{H}_{g,1}$: group of homology cobordism classes of homology cylinders over $\Sigma_{g,1}$

central extension

$$0 \to \mathbf{\Theta}^3 = \mathcal{H}_{0,1} \to \mathcal{H}_{g,1} \to \overline{\mathcal{H}}_{g,1} \to 1$$

$\mathcal{M}_{g,1} = \{(\Sigma_{g,1} \times I, \varphi); \varphi : \Sigma_{g,1} \cong \Sigma_{g,1} \times \{1\}\}$

$\mathcal{H}_{g,1} = \{(\textbf{homology}\, \Sigma_{g,1} \times I, \varphi); \varphi : \Sigma_{g,1} \cong \Sigma_{g,1} \times \{1\}\}$/homology cobordism

Strategy

Define infinitely many cohomology classes in

$$H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

which may be conjecturally equivalent to the **Euler class**

$$\chi(\mathcal{H}_{g,1}) \in H^2(\overline{\mathcal{H}}_{g,1};\boldsymbol{\Theta}^3)$$

of the above central extension

monodromy representation:

$$\mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}_0\pi_1\Sigma_{g,1}$$

is an isomorphism (Dehn-Nielsen-Zieschang), where Aut_0 denotes

symplectic automorphism group in the sense:

should preserve a particular element $\zeta \in \pi_1 \Sigma_{g,1} \cong F_{2g}$

$$\mathcal{M}_{g,1} = \{\text{monodromy of } \Sigma_{g,1}\text{-bundle over } S^1\}$$

On the other hand, a theorem of Stallings: two homologically isomorphic groups have isomorphic Mal'cev completions

$\Rightarrow$ Garoufalidis-Levine define a representation

$$\mathcal{H}_{g,1} \longrightarrow \mathrm{Aut}_0(\text{Mal'cev completion of } \pi_1 \Sigma_{g,1})$$

which has a large kernel containing $\boldsymbol{\Theta}^3$

An analysis of the above representation yields many elements in $H^*(\overline{\mathcal{H}}_{g,1}; \mathbb{Q})$

2. Symplectic automorphism groups of free nilpotent groups

$\Gamma = \pi_1 \Sigma_{g,1} \cong F_{2g} \ni \zeta = [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$

lower central series:

$$\Gamma_0 = \Gamma,\ \Gamma_1 = [\Gamma, \Gamma],\ \Gamma_2 = [\Gamma_1, \Gamma], \cdots,\ \Gamma_d = [\Gamma_{d-1}, \Gamma], \cdots$$

d-th nilpotent quotient of Γ:

$$N_d = \Gamma / \Gamma_d \quad (d = 1, 2, \cdots);\ N_1 = H = H_1(\Sigma_{g,1}; \mathbb{Z})$$

particular elements:

$\zeta_d =$ image of ζ in N_d

$\zeta_1 = 0 \in N_1 = H$ but,

$\zeta_2 = \omega_0 = \sum_{i=1}^{g} x_i \wedge y_i \in \Lambda^2 H \cong \Gamma_1 / \Gamma_2 \subset N_2$

Definition

Two subgroups of $\mathrm{Aut}\, N_d$:

$$\mathrm{Aut}'_0 N_d = \{\varphi \in \mathrm{Aut}\, N_d ; \varphi(\zeta_d) = \zeta_d\}$$

$$\mathrm{Aut}_0 N_d = p(\mathrm{Aut}'_0 N_{d+1}) \quad \text{(Garoufalidis-Levine)}$$

$p : \mathrm{Aut}\, N_{d+1} \to \mathrm{Aut}\, N_d$ natural projection.

$$\mathrm{Aut}_0 N_1 = \mathrm{Aut}_0 H \cong \mathrm{Sp}(2g, \mathbb{Z}), \quad \mathrm{Aut}'_0 N_1 \cong \mathrm{GL}(2g, \mathbb{Z})$$

projective system of groups:

$$\cdots \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow \mathrm{Aut}_0 N_{d-1} \longrightarrow \cdots \longrightarrow \mathrm{Aut}_0 N_2 \longrightarrow \mathrm{Aut}_0 N_1.$$

obtain representations:

$$\rho_d : \mathcal{M}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d, \quad \rho_\infty : \mathcal{M}_{g,1} \cong \mathrm{Aut}_0 \Gamma \longrightarrow \varprojlim_{d \to \infty} \mathrm{Aut}_0 N_d$$

induced Johnson filtration $\{\mathcal{M}_{g,1}(d)\}_d$:

$$\mathcal{M}_{g,1}(d) = \mathrm{Ker}(\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0 N_d)$$

$$\mathcal{M}_{g,1}(1) = \mathcal{I}_{g,1}\text{: Torelli group}$$

and the Johnson homomorphisms $\{\tau_d\}_d$

$$\tau_d = \rho_{d+1}|_{\mathcal{M}_{g,1}(d)} : \mathcal{M}_{g,1}(d) \longrightarrow \mathfrak{h}_{g,1}(d) \subset \mathrm{Hom}(H, \mathcal{L}_{g,1}(d+1))$$

structure of $\mathrm{Aut}_0 N_d$:

$\mathcal{L}_{g,1} = \bigoplus_{d=1}^{\infty} \mathcal{L}_{g,1}(d)$: free graded Lie algebra on $H = H_1(\Sigma_{g,1}; \mathbb{Z})$

$$\mathcal{L}_{g,1}(1) = H,\ \mathcal{L}_{g,1}(2) \cong \Lambda^2 H,\ \mathcal{L}_{g,1}(d) \cong \Gamma_{d-1}/\Gamma_d$$

$\mathfrak{h}_{g,1} = \bigoplus_{d=0}^{\infty} \mathfrak{h}_{g,1}(d)$: Lie algebra of **symplectic** derivations of $\mathcal{L}_{g,1}$

$$\mathfrak{h}_{g,1}(d) = \{D \in \mathrm{Hom}(H, \mathcal{L}_{g,1}(d+1)); D(\omega_0) = 0\}$$

Poincaré duality $H^* \cong H$ induces:

$$\mathfrak{h}_{g,1}(d) \cong \mathrm{Ker}([\ ,\ ] : H \otimes \mathcal{L}_{g,1}(d+1) \to \mathcal{L}_{g,1}(d+2))$$

Garoufalidis and Levine:

$$1 \longrightarrow \mathfrak{h}_{g,1}(d) \longrightarrow \mathrm{Aut}_0 N_{d+1} \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow 1.$$

traces

$$\mathrm{trace}(2k+1) : \mathfrak{h}_{g,1}(2k+1) \longrightarrow S^{2k+1} H$$

$$\mathrm{trace}(2k+1) \text{ vanishes on Image}\, \tau_{2k+1} \;\Rightarrow$$

Coker ρ_d becomes **larger and larger**

$$\mathrm{trace}(2k+1) : \mathfrak{h}_g(2k+1) \xrightarrow{\text{surj}} S^{2k+1}H$$

$$\mathfrak{h}_g(2k+1) \subset \mathrm{Hom}(H, \mathcal{L}_g(2k+2)) \ni (f : H \to \mathcal{L}_g(2k+2)) \mapsto$$

$$\mathrm{trace}\left(\frac{\partial f(u_i)}{\partial u_j}\right)^{\text{abel}} \in S^{2k+1}H$$

$u_1, \cdots, u_g, u_{g+1}, \cdots, u_{2g}$: symplectic basis

$\dfrac{\partial f(u_i)}{\partial u_j}$: Fox free differential

Garoufalidis-Levine, Habegger:

$\rho_d : \mathcal{M}_{g,1} \to \mathrm{Aut}_0 N_d$ extends to

$$\tilde{\rho}_d : \mathcal{H}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d \quad \text{which is } \textbf{surjective}$$

$\tilde{\rho}_\infty : \mathcal{H}_{g,1} \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d$: not injective

$$\boldsymbol{\Theta}^3 \subset \mathrm{Center}(\mathcal{H}_{g,1}) \quad \text{and} \quad \tilde{\rho}_d \equiv \text{trivial} \quad \text{on } \boldsymbol{\Theta}^3$$

$\mathrm{Image}(\tilde{\rho}_\infty) =?$

Sakasai: using the concept of acyclic closure of groups due to Levine:

constructs a representation $\rho^{\mathrm{acy}} : \mathcal{H}_{g,1} \to \mathrm{Aut}\,\Gamma^{\mathrm{acy}}$ and proved

$$\mathrm{Image}\,\rho^{\mathrm{acy}} = \{\varphi \in \mathrm{Aut}\,\Gamma^{\mathrm{acy}}; \varphi(\zeta) = \zeta\}$$

3. Rational forms of the symplectic automorphism groups

$N_d \otimes \mathbb{Q}$: Mal'cev completion of $N_d \;\Rightarrow$

$$\mathrm{Aut}\, N_d \subset \mathrm{Aut}(N_d \otimes \mathbb{Q}) \quad \text{linear algebraic group}$$

this induces

$$\mathrm{Aut}_0 N_d \subset \mathrm{Aut}_0(N_d \otimes \mathbb{Q}) \quad \text{discrete, Zariski dense, subgroup}$$

$$1 \longrightarrow \mathrm{IAut}_0 N_d \longrightarrow \mathrm{Aut}_0 N_d \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}) \longrightarrow 1.$$

$$0 \longrightarrow \mathfrak{h}_{g,1}(d) \longrightarrow \mathrm{IAut}_0 N_{d+1} \longrightarrow \mathrm{IAut}_0 N_d \longrightarrow 1$$

Proposition

$\mathrm{Aut}_0 N_d$ can be embedded into $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ as a Zariski dense subgroup

split short exact sequence:

$$1 \longrightarrow \mathrm{IAut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{Aut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \mathrm{Sp}(2g, \mathbb{Q}) \longrightarrow 1$$

$\mathrm{IAut}_0 N_d$ is mapped to $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ as a Zariski dense subgroup

Lie algebra of $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$:

$$\text{rational forms} : \mathfrak{h}_{g,1}^{\mathbb{Q}} = \mathfrak{h}_{g,1} \otimes \mathbb{Q} = \bigoplus_{k=0}^{\infty} \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$

$$\text{positive ideal} : \mathfrak{h}_{g,1}^{\mathbb{Q}+} = \bigoplus_{k=1}^{\infty} \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$$

Definition

truncated Lie algebras:

$$\mathfrak{h}^{\mathbb{Q}}_{g,1}[d] = \mathfrak{h}^{\mathbb{Q}}_{g,1}/I(d), \quad \mathfrak{h}^{\mathbb{Q}+}_{g,1}[d] = \mathfrak{h}^{\mathbb{Q}+}_{g,1}/I(d)$$

where $\quad I(d)$: ideal of derivations with degree $\geq d$

Additively, we can write

$$\mathfrak{h}^{\mathbb{Q}}_{g,1}[d] = \bigoplus_{k=0}^{d-1} \mathfrak{h}^{\mathbb{Q}}_{g,1}(k), \quad \mathfrak{h}^{\mathbb{Q}+}_{g,1}[d] = \bigoplus_{k=1}^{d-1} \mathfrak{h}^{\mathbb{Q}}_{g,1}(k).$$

Theorem 1

The Lie algebras of the algebraic groups $\mathrm{Aut}_0(N_d \otimes \mathbb{Q})$ and $\mathrm{IAut}_0(N_d \otimes \mathbb{Q})$ are isomorphic to the truncated Lie algebras

$$\mathfrak{h}_{g,1}^{\mathbb{Q}}[d], \quad \mathfrak{h}_{g,1}^{\mathbb{Q}+}[d]$$

respectively.

4. Group version of the trace maps

Theorem 1 implies the group version of the trace maps:

$$\widetilde{\mathrm{trace}}(2k+1) : \mathrm{IAut}_0 N_d \longrightarrow S^{2k+1} H_{\mathbb{Q}} \quad (2k+1 \leq d-1)$$

and we have a homomorphism

$$\widetilde{\mathrm{trace}} : \mathrm{IAut}_0 N_d \overset{i}{\subset} \mathrm{IAut}_0(N_d \otimes \mathbb{Q}) \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}$$

this can be extended to a **crossed homomorphim**

$$\widetilde{\mathrm{trace}}(2k+1) : \mathrm{Aut}_0 N_d \longrightarrow S^{2k+1} H_{\mathbb{Q}}$$

Theorem 2

For any $d \geq 2$, let ℓ denote the largest integer such that $2\ell+1 \leq d-1$. Then the group version of traces gives rise to a crossed homomorphism

$$\widetilde{\mathrm{trace}} : \mathrm{Aut}_0 N_d \longrightarrow \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}$$

Conjecture 1

The above theorem gives the abelianization of $\mathrm{IAut}_0 N_d$ modulo torsions:

$$H_1(\mathrm{IAut}_0 N_d; \mathbb{Q}) \cong \Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}}$$

5. A representation of $\mathcal{H}_{g,1}$

$\tilde{\rho}_d : \mathcal{H}_{g,1} \longrightarrow \mathrm{Aut}_0 N_d$ is *surjective* for all d, and we have a representation

$$\mathrm{Aut}_0 N_d \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\ell} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

by letting d go to the infinity, we obtain

Theorem 3

There exists a homomorphism

$$\tilde{\rho} : \mathcal{H}_{g,1} \longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

whose image is Zariski dense.

$\mathcal{IH}_{g,1} := \mathrm{Ker}(\mathcal{H}_{g,1} \longrightarrow \mathrm{Sp}(2g,\mathbb{Q}))$, Torelli homology cylinders

Corollary

The subgroup $\mathcal{IH}_{g,1}$ of $\mathcal{H}_{g,1}$ is not finitely generated because the rank of its abelianization is already infinite.

$\mathcal{I}_{g,1}$: *finitely generated* by Johnson for $g \geq 3$

Sakasai : the abelianization of the IA automorphism group $\mathrm{IAut}\, F_n^{\mathrm{acy}}$ of the acyclic closure F_n^{acy} of a free group F_n of rank $n \geq 2$ has infinite rank.

Does the homomorphism $\tilde{\rho}$ give the abelianization of the group $\mathcal{IH}_{g,1}$ modulo torsions?

6. Cohomology classes of $\mathcal{H}_{g,1}$

$\tilde{\rho}$ induces a homomorphism in cohomology:

$$\mathbb{Q}[c_1, c_3, \cdots] \otimes H^*\left(\Lambda^3 H_{\mathbb{Q}} \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}}\right)^{\mathrm{Sp}} \longrightarrow H^*(\mathcal{H}_{g,1}; \mathbb{Q})$$

Question

How non-trivial are these classes ?

restriction to $H^*(\mathcal{M}_{g,1}; \mathbb{Q})$:

$$\mathbb{Q}[c_1, c_3, \cdots] \otimes H^*(\Lambda^3 H_{\mathbb{Q}})^{\mathrm{Sp}} \longrightarrow H^*(\mathcal{H}_{g,1}; \mathbb{Q}) \longrightarrow H^*(\mathcal{M}_{g,1}; \mathbb{Q})$$

$H^*(\Lambda^3 H_{\mathbb{Q}})^{\mathrm{Sp}}$ gives rise to all the MMM classes

Kawazumi-M: image =**tautological subalgebra**

$\tilde{\rho}$ = trivial on $\mathbf{\Theta}^3 \ \Rightarrow \ \tilde{\rho}$ factors through

$$\bar{\rho} : \overline{\mathcal{H}}_{g,1} \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d$$

and we have

$$\mathbf{\Theta}^3 \longrightarrow \mathcal{H}_{g,1} \longrightarrow \overline{\mathcal{H}}_{g,1} \longrightarrow \varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d$$
$$\longrightarrow \left(\Lambda^3 H_{\mathbb{Q}} \bigoplus_{k=1}^{\infty} S^{2k+1} H_{\mathbb{Q}} \right) \rtimes \mathrm{Sp}(2g, \mathbb{Q})$$

Definition

$\tilde{\mathbf{t}}_{\mathbf{2k+1}} := \widetilde{\mathrm{trace}}^*(\iota_{2k+1}) \in H^2(\varprojlim_{d\to\infty} \mathrm{Aut}_0 N_d;\mathbb{Q})$ and consider

$$\bar{\rho}^*(\tilde{t}_{2k+1}) \in H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q})$$

where $\iota_{2k+1} \in H^2(S^{2k+1}H;\mathbb{Q})^{\mathrm{Sp}}$ denotes the generator

Main Conjecture

We have isomorphisms:

$$H^2(\overline{\mathcal{H}}_{g,1};\mathbb{Q}) \cong \mathbb{Q} < e_1, \bar{\rho}^*(\tilde{t}_3), \bar{\rho}^*(\tilde{t}_5), \cdots >$$

$$H^2(\mathcal{H}_{g,1};\mathbb{Q}) \cong \mathbb{Q} < e_1 >$$

$H^2(\mathcal{M}_{g,1};\mathbb{Q}) \cong \mathbb{Q} < e_1 >$ by Harer

Lie algebra version: $t_{2k+1} = \mathrm{trace}(2k+1)^*(\iota_{2k+1}) \in H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})_{4k+2}$

Main Conjecture, Lie algebra case

We have an isomorphism:

$$H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})^{\mathrm{Sp}} \cong \mathbb{Q} < e_1, t_3, t_5, \cdots >$$

via a theorem of Kontsevich:

$t_{2k+1} \;\Leftrightarrow\; \mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}; \mathbb{Q}) \quad \text{and} \quad t_{2k+1} \neq 0 \;\Leftrightarrow\; \mu_k \neq 0$

M.: $t_3 \neq 0$ Conant and Vogtmann: $\mu_2 \neq 0$

$$H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})_6 \cong \mathbb{Q} \text{ (Hatcher-Vogtmann)}, \quad H^2(\mathfrak{h}_{g,1}^{\mathbb{Q}})_{10} \cong \mathbb{Q} \text{ (Ohashi)}$$

unknown for $t_{2k+1}, \mu_k \ (k \geq 3)$

If true $\Rightarrow$ obtain series of homomorphisms

$$\hat{t}_{2k+1} \longrightarrow \mathbf{\Theta}^3 \quad (k = 1, 2, \cdots)$$

Related difficult questions:

1. Is $\mathcal{H}_{g,1}$ **perfect** ? like in the case of $\mathcal{M}_{g,1}$

2. Does $H^*(\mathcal{H}_{g,1};\mathbb{Q})$ stabilize ? like in the case of $H^*(\mathcal{M}_{g,1};\mathbb{Q})$

Harer stability theorem

3. Does the Grothendieck Riemann-Roch theorem (or the Atiyah-Singer index theorem for families), applied to the Chern classes of the Hodge bundle and the MMM-classes of odd indices, continue to hold in the setting of **homological** surface bundles ? By Theorem 3 and its consequence, both classes are defined as elements of $H^*(\mathcal{H}_{g,1};\mathbb{Q})$.