On the Magnus representation of the mapping class group

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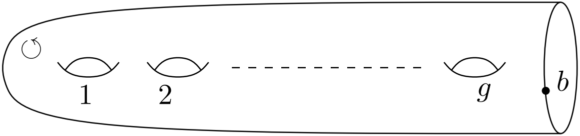
March 25, 2008

$\Sigma_{g,1} =$



$$\mathcal{M}_{g,1} = \pi_0\left(\mathrm{Diff}_+(\Sigma_{g,1}, \partial\Sigma_{g,1})\right)$$

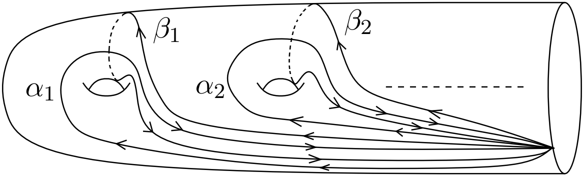
Mapping class group

$$\mathcal{I}_{g,1} = \{\varphi \in \mathcal{M}_{g,1} \mid \varphi \text{ acts trivially on } H = H_1(\Sigma_{g,1};\mathbb{Z})\}$$

Torelli group

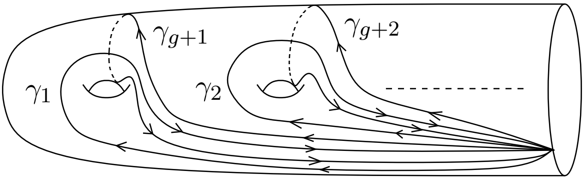
$$1 \longrightarrow \mathcal{I}_{g,1} \longrightarrow \mathcal{M}_{g,1} \longrightarrow Sp(2g;\mathbb{Z}) \longrightarrow 1$$

$\Gamma_0 = \pi_1(\Sigma_{g,1}, b)$ is a free group generated by α_i, β_i.



$$\gamma_i = \alpha_i, \quad \gamma_{g+i} = \beta_i \qquad (i = 1, \ldots, g)$$

$\Gamma_0 = \pi_1(\Sigma_{g,1}, b)$ is a free group generated by $\gamma_1, \ldots, \gamma_{2g}$.



$$\gamma_i = \alpha_i, \quad \gamma_{g+i} = \beta_i \qquad (i = 1, \ldots, g)$$

Theorem. (Dehn, Nielsen, Zieschang)

$$\mathcal{M}_{g,1} = \left\{ \varphi \in \mathrm{Aut}\Gamma_0 \;\middle|\; \varphi\left(\prod_{i=1}^{g}[\alpha_i, \beta_i]\right) = \prod_{i=1}^{g}[\alpha_i, \beta_i] \right\}$$

Definition.

The following mapping r is called the Magnus representation of the mapping class group

$$\begin{array}{rccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g;\mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \left(\overline{\dfrac{\partial \varphi(\gamma_j)}{\partial \gamma_i}}\right)_{i,j} \end{array}$$

where $\frac{\partial}{\partial \gamma_i}$ is the Fox derivation ▸ Fox derivation and $^{-}$ is the mapping induced by the antiautomorphism $\gamma \mapsto \gamma^{-1}$.

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This mapping r is *injective*.

$$r : \mathcal{M}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[\Gamma_0])$$

Proposition. (Morita)

For $\varphi, \psi \in \mathcal{M}_{g,1}$,

$$r(\varphi\psi) = r(\varphi) \cdot {}^{\varphi}r(\psi)$$

Here ${}^{\varphi}r(\psi)$ denotes the matrix obtained from $r(\psi)$ by applying the automorphism $\varphi : \mathbb{Z}[\Gamma_0] \to \mathbb{Z}[\Gamma_0]$ on each entry.

Γ_k : the k-th term of the lower central series of Γ_0.
i.e. $\Gamma_{k+1} = [\Gamma_k, \Gamma_0]$

$N_k = \Gamma_0/\Gamma_k$: the k-th nilpotent quotient
$\phi_k : \Gamma_0 \to N_k$: the projection

$$\mathcal{M}_{(k)} = \{\varphi \in \mathcal{M}_{g,1} \mid \varphi \text{ acts trivially on } N_k\}$$

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Example. $k = 1$

$N_1 = \Gamma_0/[\Gamma_0, \Gamma_0] = H$

$$\begin{array}{cccccl} \phi_1 : & \Gamma_0 & \longrightarrow & H & : & \text{abelianization} \\ & \alpha_i & \longmapsto & x_i & & \\ & \beta_i & \longmapsto & y_i & & \end{array}$$

$\mathcal{M}_{(1)} = \mathcal{I}_{g,1}$: Torelli group

graded Magnus representation

$$\begin{array}{ccc} \mathcal{M}_{g,1} & \xrightarrow{r} & GL(2g;\mathbb{Z}[\Gamma_0]) \\ \cup & & \downarrow \phi_k \\ \mathcal{M}_{(k)} & \xrightarrow{r_k} & GL(2g;\mathbb{Z}[N_k]) \end{array}$$

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r_k is a genuine representation.

Proof.

Since $\mathcal{M}_{(k)}$ acts trivially on $\mathbb{Z}[N_k]$. For $\varphi, \psi \in \mathcal{M}_{(k)}$,

$$r_k(\varphi\psi) = r_k(\varphi) \cdot r_k(\psi)$$

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Magnus representation of the Torelli group

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g;\mathbb{Z}[H])$$

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$$\varphi : \quad \Sigma_{g,1} \longrightarrow \Sigma_{g,1} \quad : \quad \text{diffeomorphism}$$

$$\rightsquigarrow \quad \widetilde{\varphi} : \quad \widetilde{\Sigma} \longrightarrow \widetilde{\Sigma} \quad : \quad \text{diffeomorphism}$$

(fix the fiber over b pointwise)

$$\rightsquigarrow \quad \widetilde{\varphi}_* : \quad H_1(\widetilde{\Sigma}, p^{-1}(b); \mathbb{Z}) \longrightarrow H_1(\widetilde{\Sigma}, p^{-1}(b); \mathbb{Z})$$

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The covering transformation group is Γ_0.
We fix a lift of b to $\widetilde{\Sigma}$ denoted by $\widetilde{b}$.

γ_i : a generator for Γ_0 $\quad (i = 1, 2, \ldots, 2g)$
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$H_1(\widetilde{\Sigma}, p^{-1}(b); \mathbb{Z})$:
a free $\mathbb{Z}[\Gamma_0]$-module with free basis $[\widetilde{\gamma}_1], [\widetilde{\gamma}_2], \ldots, [\widetilde{\gamma}_{2g}]$.

$$\mathcal{M}_{g,1} \text{ acts on } H_1(\widetilde{\Sigma}, p^{-1}(b); \mathbb{Z}) \simeq \mathbb{Z}[\Gamma_0]^{2g}$$

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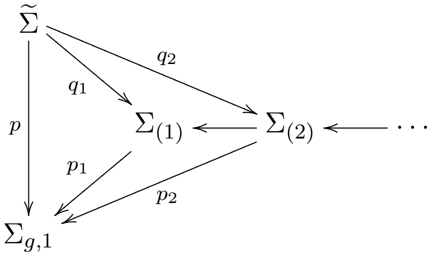
Geometric interpretation of the Magnus representation

$$\begin{array}{cccc} r: & \mathcal{M}_{g,1} & \longrightarrow & GL(2g; \mathbb{Z}[\Gamma_0]) \\ & \varphi & \longmapsto & \overline{{}^t\widetilde{\varphi}_*} \end{array}$$

where $^-$ is a mapping induced by $\gamma \mapsto \gamma^{-1}$.

$$\Gamma_{k+1} = [\Gamma_k, \Gamma_0], \quad N_k = \Gamma_0/\Gamma_k, \quad \phi_k : \Gamma_0 \to N_k$$

$p_k : \Sigma_{(k)} \longrightarrow \Sigma_{g,1}$: the regular covering corresponding to $\ker \phi_k$
$q_k : \widetilde{\Sigma} \longrightarrow \Sigma_{(k)}$: the projection



Example.
$\phi_1 : \Gamma_0 \to H$: abelianization
$p_1 : \Sigma_{(1)} \to \Sigma_{g,1}$ is the universal abelian covering

$H_1(\Sigma_{(k)}, p_k^{-1}(b); \mathbb{Z})$:
a free $\mathbb{Z}[N_k]$-module with free basis $[q_k(\widetilde{\gamma}_1)], [q_k(\widetilde{\gamma}_2)], \ldots, [q_k(\widetilde{\gamma}_{2g})]$.

$$\mathcal{M}_{(k)} \text{ acts on } H_1(\Sigma_{(k)}, p_k^{-1}(b); \mathbb{Z}) \simeq \mathbb{Z}[N_k]^{2g}$$

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$$r_k : \mathcal{M}_{(k)} \longrightarrow GL(2g; \mathbb{Z}[N_k])$$

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$$r_k : \mathcal{M}_{(k)} \longrightarrow GL(2g; \mathbb{Z}[N_k])$$

Example.

$$\mathcal{I}_{g,1} \text{ acts on } H_1(\Sigma_{(1)}, p_1^{-1}(b); \mathbb{Z}) \simeq \mathbb{Z}[H]^{2g}$$

$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[H])$$

The Magnus representation of the Torelli group

Theorem. (Morita)

$$\overline{r_k} : \mathcal{M}_{(k)} \xrightarrow{r} GL(2g;\mathbb{Z}[\Gamma_0]) \xrightarrow{/I^k} GL(2g;\mathbb{Z}[\Gamma_0]/I^k)$$

$$\mathcal{M}_{(k)} \xrightarrow{\tau_k} \mathrm{Hom}(H, \Gamma_k/\Gamma_{k+1}) \xrightarrow{\|\cdot\|} M(2g, I^k/I^{k+1})$$

$$\overline{r_k}(\varphi) = I_{2g} + \overline{\|\tau_k(\varphi)\|} \qquad \text{for any } \varphi \in \mathcal{M}_{(k)}$$

where

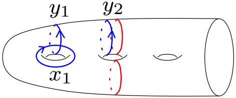
I : the augmentation ideal of $\mathbb{Z}[\Gamma_0]$

τ_k : the k-th Johnson homomorphism

$\|f\| = \left(\frac{\partial f([\gamma_j])}{\partial \gamma_i}\right)_{i,j}$ for $f \in \mathrm{Hom}(H, \Gamma_k/\Gamma_{k+1})$

I_{2g} : the identity matrix

Example. $\varphi_1 \in \mathcal{I}_{3,1}$: *standard* genus 1 BP map



$$\begin{array}{cccc} \tau_1 : & \mathcal{I}_{3,1} & \longrightarrow & \wedge^3 H \\ & \varphi_1 & \longmapsto & x_1 \wedge y_1 \wedge y_2 \end{array}$$

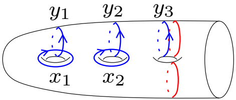
1-st Johnson hom.

$$\overline{r_1}(\varphi_1) = I_6 + \overline{\|\tau_1(\varphi_1)\|}, \qquad \|\tau_1(\varphi_1)\| \in M(6, I/I^2) \simeq M(6, H)$$

$u_1, u_2, u_3, v_1, v_2, v_3$: generators of $I/I^2 \simeq H$

$$\overline{r_1}(\varphi_1) = I_6 + \begin{pmatrix} -v_2 & v_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -u_1 & 0 & -v_2 & 0 & 0 \\ u_1 & 0 & 0 & v_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Example. $\varphi_2 \in \mathcal{I}_{3,1}$: *standard* genus 2 BP map



$$\begin{array}{llll} \tau_1 : & \mathcal{I}_{3,1} & \longrightarrow & \wedge^3 H \\ & \varphi_2 & \longmapsto & x_1 \wedge y_1 \wedge y_3 + x_2 \wedge y_2 \wedge y_3 \end{array}$$

1-st Johnson hom.

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$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[H])$$

Theorem. (S. 2003)

For $g \geq 2$ there exists a non-singular matrix $P \in GL(2g; R)$ such that for any element $\varphi \in \mathcal{I}_{g,1}$

$$P^{-1} r_1(\varphi) P = \left(\begin{array}{c|ccc|c} 1 & & * & & * \\ \hline 0 & & & & \\ \vdots & & \rho_B(\varphi) & & * \\ & & & & \\ \cline{2-5} 0 & 0 & \cdots & 0 & 1 \end{array}\right).$$

Moreover, ρ_B is a $(2g-2)$-dimensional irreducible representation of $\mathcal{I}_{g,1}$. Here $R = \mathbb{Z}[{x_i}^{\pm 1}, {y_i}^{\pm 1}, \frac{1}{1-y_i}]$ ($\supset \mathbb{Z}[H]$)

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Theorem. (S. 2002)

The Magnus representation of the Torelli group is not faithful for $g \geq 2$.

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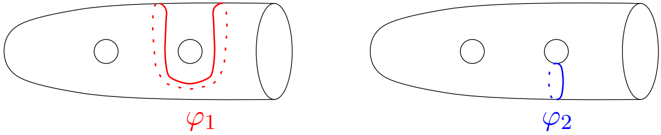
Example. (2000)



$$[\varphi_1, \varphi_2 \varphi_1 \varphi_2^{-1}] \in \ker r_1$$

Definition.

c_1, c_2 : oriented bounding simple closed curves on $\Sigma_{g,1}$

$$\langle c_1, c_2 \rangle_H = \sum_{h \in H} (h\widehat{c_1}, \widehat{c_2})\, h \quad \in \mathbb{Z}[H]$$

$\widehat{c_1}, \widehat{c_2}$: the lifts of c_1, c_2 to the universal abelian covering $\Sigma_{(1)}$
$(\,\cdot\,,\,\cdot\,)$: the algebraic intersection number

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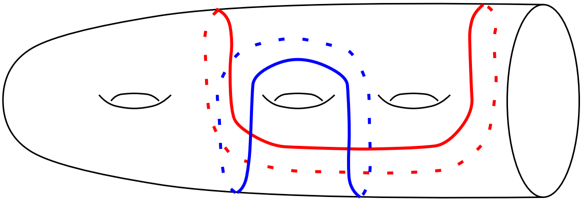
Proposition. (S. 2006)

c_1, c_2 : two bounding simple closed curves
T_{c_1}, T_{c_2} : the Dehn twists along c_1, c_2 respectively.

$$\langle c_1, c_2 \rangle_H = 0 \Longrightarrow [T_{c_1}, T_{c_2}] \in \ker r_1$$

Example.



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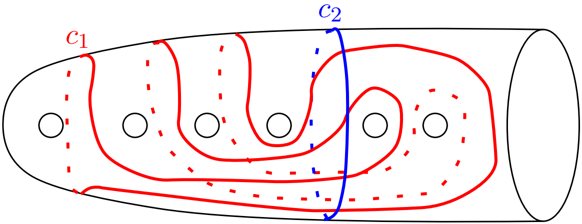

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Corollary.

c_1, c_2 : two bounding simple closed curves

$$i(c_1, c_2) = 2 \implies [T_{c_1}, T_{c_2}] \in \ker r_1$$

where $i(\cdot\,,\,\cdot)$ is the geometric intersection number

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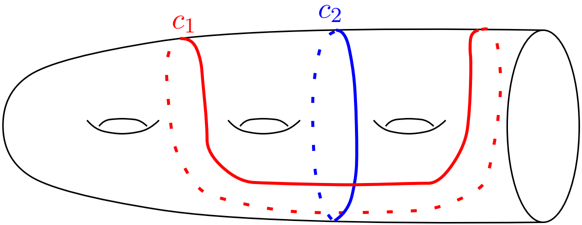

$$[T_{c_1}, T_{c_2}] \in \ker r_1$$

Theorem. (S. 2005)

c_1, c_2 : two bounding simple closed curves

$$[T_{c_1}, T_{c_2}] \in \ker r_1 \iff \det(\lambda I_{2g} - r_1(T_{c_1} T_{c_2})) = (\lambda - 1)^{2g}$$

Theorem. (S. 2005)

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Remark.

c : a bounding simple closed curve

$$\det(\lambda I_{2g} - r_1(T_c)) = (\lambda - 1)^{2g}$$

Proof. (outline)

$$-\langle c_1, c_2 \rangle_H \cdot \langle c_2, c_1 \rangle_H = \mathrm{tr}(I_{2g} - r_1(T_{c_1}T_{c_2}))$$

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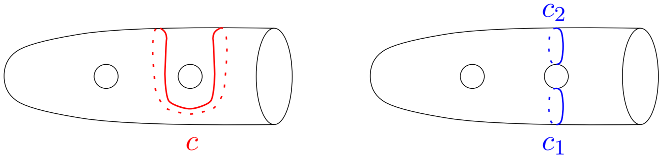

$$T_{c_1} T_{c_2}^{-1} \in G \Longrightarrow [T_c,\ T_{c_1} T_{c_2}^{-1} T_c (T_{c_1} T_{c_2}^{-1})^{-1}] \in \ker r_1$$

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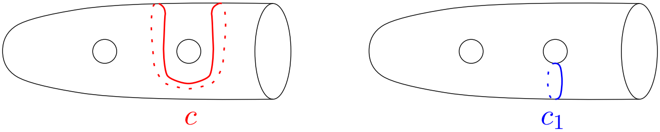

$$[T_c,\ T_{c_1} T_c T_{c_1}^{-1}] \in \ker r_1$$

$G = \ker(\mathcal{I}_{g,1} \longrightarrow \mathcal{I}_g)$: the kernel of *disc-filling homomorphism*

$$G^{(0)} = G, \qquad G^{(k+1)} = [G^{(k)}, G^{(k)}]$$
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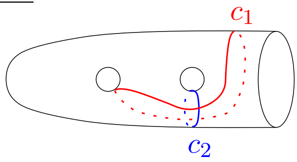
Theorem. (S.)

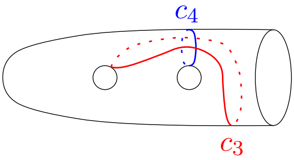
- $G^{(3)} \subset \ker r_1$, $\qquad [G^{(2)}, \mathcal{I}_{g,1}] \subset \ker r_1$
- $G_{(n)} \not\subset \ker r_1$ for any n

Remark. (S. 2003)

- $\mathcal{I}_{(n)} \not\subset \ker r_1$ for any n
  where $\mathcal{I}_{(0)} = \mathcal{I}_{g,1}$, $\mathcal{I}_{(k+1)} = [\mathcal{I}_{(k)}, \mathcal{I}_{g,1}]$

Example. (2007)





$$[[T_{c_1}, T_{c_2}], [T_{c_3}, T_{c_4}]] \in \ker r_1$$

$$[T_{c_1}, T_{c_2}], [T_{c_3}, T_{c_4}] \in G^{(1)} \implies [[T_{c_1}, T_{c_2}], [T_{c_3}, T_{c_4}]] \in G^{(2)}$$

Problem.

Is the graded Magnus representation faithful for $k \geq 2$?

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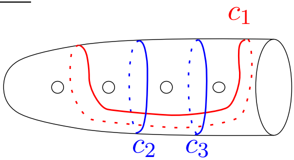
$$[\ker r_1, \ker r_1] \not\subset \ker r_2 \ ?$$

Example.



$$[T_{c_1}, T_{c_2}], [T_{c_1}, T_{c_3}] \in \ker r_1$$

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$$r_1 : \mathcal{I}_{g,1} \longrightarrow GL(2g; \mathbb{Z}[H])$$

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Is the kernel of r_1 finitely generated?

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Theorem. (Biss-Farb, 2006)

$\mathcal{K}_g$ is not finitely generated.
(Then $\mathcal{K}_{g,1} (= \mathcal{M}_{(2)})$ is not finitely generated.)

They constructed a series of elements of $\mathcal{K}_g$ satisfying *a certain condition*.

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Problem. (Morita)

$$d_1 : \mathcal{K}_g \longrightarrow \mathbb{Z}, \quad d_1 \text{ is a generator of } H^1(\mathcal{K}_g, \mathbb{Z})^{\mathcal{M}_g}$$

Determine whether r_1 detects d_1 or not. In other words, does there exist some $\varphi \in \ker r_1$ so that $d_1(\varphi) \neq 0$.

Remark.

- $\mathcal{K}_{g,1} \longrightarrow \mathcal{K}_g$: disc-filling
- $\ker r_1 \subset \mathcal{K}_{g,1}$
- $\varphi \in [\mathcal{K}_{g,1}, \mathcal{K}_{g,1}] \Longrightarrow d_1(\varphi) = 0$

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Does there exists a pair of BP maps φ_1, φ_2 so that $[\varphi_1, \varphi_2] \in \ker r_1$?

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Theorem. (Johnson)

$\mathcal{I}_{g,1}$ is generated by genus 1 BP maps for $g \geq 3$.

Proposition. (S. 2007)

For $g = 2$,

φ_1 : genus 1 BP map

$$[\varphi_1, \psi\varphi_1\psi^{-1}] \notin \ker r_1 \qquad \text{for any } \psi \in \mathcal{I}_{2,1}$$

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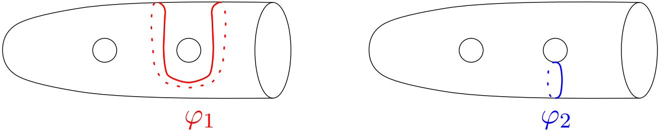
$$\ker r_1 \subset \ker(\mathcal{I}_{2,1} \to \mathcal{I}_2)? \quad \text{for } g = 2$$



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$$\ker r_1 \not\subset \ker(\mathcal{I}_{g,1} \to \mathcal{I}_g) \quad \text{for } g \geq 3$$

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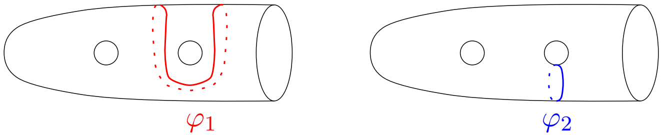

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Definition.

$$\frac{\partial}{\partial x_j} : \mathbb{Z}[F_u] \longrightarrow \mathbb{Z}[F_u] : \text{ the Fox derivation}$$

$$\begin{aligned} \frac{\partial}{\partial x_j}(x_{\mu_1}^{\varepsilon_1} \cdots x_{\mu_r}^{\varepsilon_r}) &= \sum_{i=1}^{r} \varepsilon_i \, \delta_{\mu_i, j} \, x_{\mu_1}^{\varepsilon_1} \cdots x_{\mu_{i-1}}^{\varepsilon_{i-1}} x_{\mu_i}^{\frac{1}{2}(\varepsilon_i - 1)}, \\ \frac{\partial}{\partial x_j}\left(\sum a_w \, w\right) &= \sum a_w \frac{\partial w}{\partial x_j}, \end{aligned}$$

$$\varepsilon_i = \pm 1, \; w \in F_u, \; a_w \in \mathbb{Z}.$$

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