Multicurve Cohomology of Mapping Class Groups

Rasmus Villemoes

Center for the Topology and Quantization of Moduli spaces

CTQM Workshop, March 27, 2008

or more precisely

The first cohomology group of mapping class groups with coefficients in formal linear combinations of multicurves

Rasmus Villemoes

Center for the Topology and Quantization of Moduli spaces

CTQM Workshop, March 27, 2008

References

Joint with J. Andersen.

- arXiv:0710.2203
- arXiv:0802.3000
- arXiv:0802.4372

Basic setup

- Σ is an oriented genus g surface with one puncture
- $\Gamma = \pi_0(\mathrm{Diff}^+(\Sigma))$ is the mapping class group of Σ
- $\pi_1 = \pi_1(\Sigma)$ is the fundamental group of Σ.

The $\mathrm{SU}(2)$ moduli space is by definition

$$\mathcal{M}_{\mathrm{SU}(2)} = \{\varrho \colon \pi_1 \to \mathrm{SU}(2)\}/\mathrm{SU}(2).$$

The mapping class group acts on $\mathcal{M}_{\mathrm{SU}(2)}$ via the outer action on π_1.

A symplectic manifold

Let γ be a small loop winding once around the puncture. Then inside $\mathcal{M}_{\mathrm{SU}(2)}$ there is a subspace

$$\mathcal{M}' = \{\varrho\colon \pi_1 \to \mathrm{SU}(2) \mid \varrho(\gamma) = -I\}/\mathrm{SU}(2)$$

This is clearly preserved by the action of Γ.

Fact

The set $\mathcal{M}'$ is a smooth symplectic manifold. The mapping class group acts by symplectomorphisms.

Poisson structure

Symplectic manifolds are also Poisson manifolds, meaning there exists a Lie bracket

$$\{-,-\}\colon C^\infty(\mathcal{M}') \times C^\infty(\mathcal{M}') \to C^\infty(\mathcal{M}').$$

This satifies $\{fg, h\} = f\{g, h\} + \{f, h\}g$.
The mapping class group action on $C^\infty(\mathcal{M}')$ is by Lie algebra homomorphisms.

Deformation quantization

Definition

A star product on $\mathcal{M}'$ is an associative h-bilinear product

$$*\colon C^\infty(\mathcal{M}')[[h]] \times C^\infty(\mathcal{M}')[[h]] \to C^\infty(\mathcal{M}')[[h]]$$

satisfying

- $f * g = fg \mod h$ and
- $(f * g - g * f)/h = c\{f, g\} \mod h$

for any $f, g \in C^\infty(\mathcal{M}')$.

A star product is a "deformation" of the usual commutative multiplication of formal power series "in the direction of" the Poisson bracket.

Equivalence of star products

Two star products $*, *'$ are equivalent if there exists a linear map

$$T = \mathrm{Id} + \sum_{j=1}^{\infty} h^j T_j \colon C^\infty(\mathcal{M}')[[h]] \to C^\infty(\mathcal{M}')[[h]]$$

intertwining them, ie. satisfying $T(f * g) = T(f) *' T(g)$

Philosophy

The symplectic and hence Poisson structures on $\mathcal{M}'$ are mapping class group invariant.
Hence two questions are natural:

- Does there exist Γ-invariant star products?
- To what extent is such a star product unique?

The latter question is partially answered by

Theorem (Andersen)

Provided $H^1(\Gamma, C^\infty(\mathcal{M}')) = 0$, any two equivalent Γ-equivariant star products are identical.

Obvious question

Theorem (Andersen)

Provided $H^1(\Gamma, C^\infty(\mathcal{M}')) = 0$, *any two equivalent* Γ*-equivariant star products are identical.*

Question

Does $H^1(\Gamma, C^\infty(\mathcal{M}'))$ *vanish?*

This is too hard to handle directly, at least for me.

Changing module

Let $\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$ denote the $\mathrm{SL}_2(\mathbb{C})$ moduli space. This is an affine algebraic set, so we may let $\mathcal{O} = \mathcal{O}(\mathcal{M}_{\mathrm{SL}_2(\mathbb{C})})$ denote the space of regular functions.

Remark

Notice that $\mathcal{M}' \subset \mathcal{M}_{\mathrm{SU}(2)} \subset \mathcal{M}_{\mathrm{SL}_2(\mathbb{C})}$. *The latter inclusion is non-trivial but easy.*

Main Theorem

For $g \geq 2$, *we have* $H^1(\Gamma, \mathcal{O}) = 0$.

Translating the coefficients

Main Theorem

For $g \geq 2$, we have $H^1(\Gamma, \mathcal{O}) = 0$.

We will prove this in two steps:

- Identify $\mathcal{O}$ with another space which is easier to work with.
- Make and prove a more general statement.

Multicurves

Definition

A *multicurve* is the homotopy class of an embedding

$$\bigsqcup_n S^1 \to \Sigma,$$

such that no component bounds a disc. (Informally, a finite number of disjoint non-trivial circles on Σ, or a closed 1-submanifold).

We let S denote the set of multicurves in Σ. Clearly Γ acts on S. Let $\mathbb{C}S$ denote the complex vector space freely spanned by S.

An isomorphism

Theorem (Bullock, Frohman, Kania-Bartoszyńska; Skovborg)

There is a Γ-isomorphism $\mathbb{C}S \to \mathcal{O}$.

This isomorphism is inspired by Goldman's notion of holonomy functions on the moduli space.
Clearly the Main Theorem is now equivalent to the vanishing of $H^1(\Gamma, \mathbb{C}S)$.

Generalizing the Main Theorem

In fact, we prove a somewhat stronger theorem.
Let A be any abelian group, and let

$$AS = \mathrm{Map}_f(S, A)$$

denote the set of all finite formal A-combinations of elements of S, or equivalently the set of maps $S \to A$ which vanish for all but finitely many $e \in S$.

Theorem

If A is torsion-free, $H^1(\Gamma, AS) = 0$.

Splitting the coefficients

Let $R \subset S$ denote a set of representatives of the Γ-orbits of S. For $e \in R$, let

$$M_e = A(\Gamma e) = \mathrm{Map}_f(\Gamma e, A) \qquad \text{and} \qquad \hat{M}_e = \mathrm{Map}(\Gamma e, A).$$

Clearly we have decompositions

$$AS = \bigoplus_{e \in R} M_e \tag{1}$$

and $\mathrm{Map}(S, A) = \prod_{e \in R} \hat{M}_e$ as Γ-modules.

Splitting the cohomology

The splitting (1) yields

$$H^1(\Gamma, AS) = \bigoplus_{e \in R} H^1(\Gamma, M_e)$$

and we must prove that

$$H^1(\Gamma, M_e) = 0$$

for each $e \in R$.

Remark

We have $|\Gamma e| = \infty$ unless $e = \emptyset$. In that case $H^1(\Gamma, M_\emptyset) = H^1(\Gamma, A) = \mathrm{Hom}(\Gamma, A) = 0$, at least if $g \geq 3$ or if $g = 2$ and A has no 2 and 5-torsion.

Exact sequences

The short exact sequence $0 \longrightarrow M_e \longrightarrow \hat{M}_e \longrightarrow \hat{M}_e / M_e \longrightarrow 0$ of Γ-modules induces a long exact cohomology sequence

$$H^0(\Gamma, \hat{M}_e) \xrightarrow{p} H^0(\Gamma, \hat{M}_e / M_e) \longrightarrow H^1(\Gamma, M_e) \xrightarrow{i} H^1(\Gamma, \hat{M}_e)$$

(1) Describe/compute $H^1(\Gamma, \hat{M}_e)$.

(2) Use (1) to prove that $i = 0$.

(3) Prove that p is surjective, so that i is injective.

If the zero map is injective, its domain must be zero!

Step (1): Computing $H^1(\Gamma, \hat{M}_e)$

Let $\Gamma_e \subseteq \Gamma$ denote the stabilizer of e. Then a theorem from the standard group cohomological toolbox, known as Shapiro's Lemma, gives an isomorphism

$$H^1(\Gamma, \hat{M}_e) \cong H^1(\Gamma_e, A) \cong \mathrm{Hom}(\Gamma_e, A)$$

The isomorphism is given explicitly as follows: A cocycle $u\colon \Gamma \to \hat{M}_e = \mathrm{Map}(\Gamma e, A)$ is mapped to

$$\mathrm{ev}_e \circ u_| : \Gamma_e \to \hat{M}_e \to A$$

In other words, one restricts u to the stabilizer of e and then picks out the coefficient of e.

Step (2): Proving that $H^1(\Gamma, M_e) \to H^1(\Gamma, \hat{M}_e)$ is zero

Let $u\colon \Gamma \to M_e$ be a cocycle. We must prove that, for any $f \in \Gamma_e$, the coefficient of e in $u(f)$ is zero.
First we handle a rather generic case, which almost suffices:

Step (2): A generic case

Asssume that α is a SCC such that the twist τ_α lies in Γ_e, and that some component of e is not a parallel copy of α. Find a SCC β disjoint from α such that $\tau_\beta e \neq e$. Then τ_α and τ_β commute, implying

$$u(\tau_\alpha \tau_\beta) = u(\tau_\alpha) + \tau_\alpha u(\tau_\beta) = u(\tau_\beta \tau_\alpha) = u(\tau_\beta) + \tau_\beta u(\tau_\alpha)$$

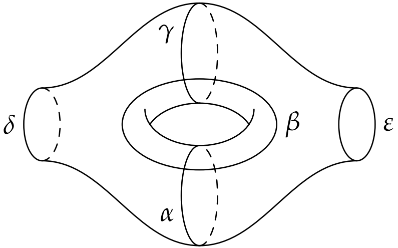
which we rewrite as

$$(1 - \tau_\beta) u(\tau_\alpha) = (1 - \tau_\alpha) u(\tau_\beta). \tag{2}$$

Now, the coefficient of e on the RHS of (2) is 0. Hence, if $u(\tau_\alpha)$ contains the non-zero term ae, it must also contain the term $a\tau_\beta^{-1}e$. By induction it contains $a\tau_\beta^{-n}e$ for all $n = 1, 2, \ldots$. This contradicts the assumption that u took values in M_e.

Step (2): Using relations in Γ

Now assume that e is simply a number of parallel copies of the SCC ε. Any such ε can be realized as the ε in a subsurface of genus 1 with two boundary components as below:



Now, the chain relation states that $(\tau_\alpha \tau_\beta \tau_\gamma)^4 = \tau_\delta \tau_\varepsilon$. Applying the cocycle condition to this easily implies that the coefficient of e in $u(\tau_\varepsilon)$ is zero.

Step (2): Final remarks

Now for the general case: Let $f \in \Gamma_e$. Since $\widetilde{u} = \mathrm{ev}_e \circ u_|$ is a homomorphism and A is torsion-free, it suffices to prove that $\widetilde{u}(f^N) = 0$ for some large N.

Choose N so large that f^N fixes each component of e. Then f^N can (though not unambigously) be thought of as a diffeomorphism of the surface obtained by cutting along e.

Hence f^N is isotopic to a product of Dehn twists which are all disjoint from e. $\square$

Step (3): Almost invariant colorings

Some terminology: Let G be a group, X a G-set (that is, a set equipped with a transitive action of G) and C a set.

- A (C-)coloring of X is a map $c\colon X \to C$.
- A coloring is *almost invariant* if, for each $g \in G$, the identity $c(x) = c(gx)$ fails for only finitely many $x \in X$.
- Two colorings are *equivalent* if they assign different colors to only finitely many elements of X; this is clearly an equivalence relation on the set of C-colorings.
- A coloring is *trivial* if it is equivalent to a monochromatic (constant) coloring.

Step (3): Colorings of multicurves

If we apply the above to the situation $G = \Gamma$, $X = \Gamma e$ we have

Proposition

For $g \geq 2$, any almost invariant coloring of Γe is trivial.

Recall that $H^0(G, N) = N^G$, the G-invariant elements of N. Hence an element of $H^0(\Gamma, \hat{M}_e / M_e)$ is represented by an element $m \in \mathrm{Map}(\Gamma e, A)$ such that $m - gm \in M_e$ for each $g \in \Gamma$. Hence, $m(x) = m(g^{-1}x)$ for all but finitely many $x \in \Gamma e$, so m is precisely an almost invariant A-coloring of Γe.
By the above proposition, m is almost constant and hence in the image of

$$H^0(\Gamma, \hat{M}_e) \to H^0(\Gamma, \hat{M}_e / M_e).$$

Step (3): Proving the coloring theorem

Let $c\colon \Gamma e \to C$ be an almost invariant coloring. The crucial observation is that if τ_α is a Dehn twist and $x \in \Gamma e$ such that $\tau_\alpha x \neq x$, then $c(\tau_\alpha^n x)$ stabilizes for large enough n.
This stable color is the future of (τ_α, x). Similarly one defines the past of such an "interesting pair".

Lemma

The future equals the past.

More explicitly, there exists an N such that for all $n, m \geq N$, we have $c(\tau_\alpha^{-m} x) = c(\tau_\alpha^n x)$.

Step (3): Proof of "fate" lemma

Since $g \geq 2$, there exists a SCC β disjoint from α which also makes an interesting pair with x, ie. $\tau_\beta x \neq x$.
Then $\{\tau_\alpha^a \tau_\beta^b x\}$, $a, b \in \mathbb{Z}$, is an $\mathbb{Z}^2$-indexed family of distinct multicurves. By the almost invariance, outside some bounded region in $\mathbb{Z}^2$ the color of a multicurve does not change by moving up, down, left or right. The future and past of (τ_α, x) can be connected by such moves. $\square$

Step (3)

We have in fact already proved the $i = 0$ case of the following lemma:

Lemma

Assume that α *and* β *are SCCs with* $i(\alpha, \beta) \leq 1$*, such that* (τ_α, x) *and* (τ_β, x) *are both interesting. Then* $\mathrm{fut}(\tau_\alpha, x) = \mathrm{fut}(\tau_\beta, x)$*.*

The proof of the $i = 1$ case is essentially just a reduction to the $i = 0$ case.

The above lemmas (and a few more) together with known generating sets for the mapping class group suffice to prove the proposition.

Comments

The "multicurve functions" take real values on the $\mathrm{SU}(2)$ moduli space, being defined using the trace. Hence one may consider the real subspace $\mathbb{R}S \subset \mathcal{O}$, and the restriction map

$$r\colon \mathbb{R}S \to C^\infty(\mathcal{M}')$$

At least two questions are relevant:

- Is r injective?
- Is the image of r dense?

Moreover, one may ask

- Would affirmative answers to any of the above help answer the original question?