ON CONFIGURATION SPACE INTEGRAL OF SMOOTH SPHERE BUNDLES

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1. INTRODUCTION

FUNDAMENTAL PROBLEM:

- Classification of smooth M-bundles, or
- Determine the homotopy type of $B\mathrm{Diff}(M)$
  $(\mathrm{Diff}(M) = \{\varphi : M \to M, C^\infty\text{-diffeom}\};\ C^\infty\text{-topology})$.

REMARK

$$\{\text{smooth } M\text{-bundles over } B\}/\text{isom} \xleftrightarrow{\text{bijec}} [B, B\mathrm{Diff}(M)]$$

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(K. Igusa) The extra $\mathbf{Q}$ can be detected by "higher FR torsion".

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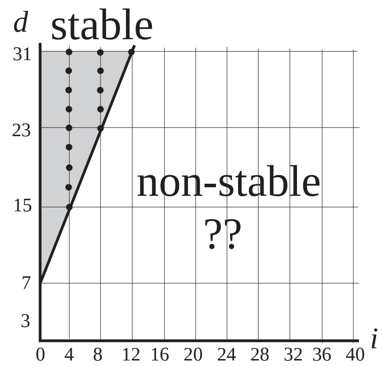

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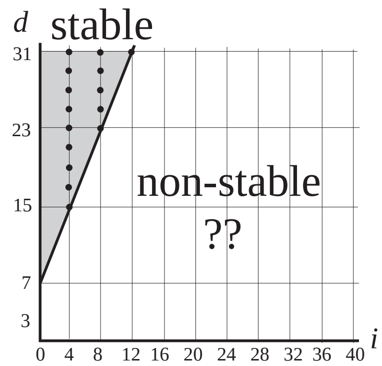

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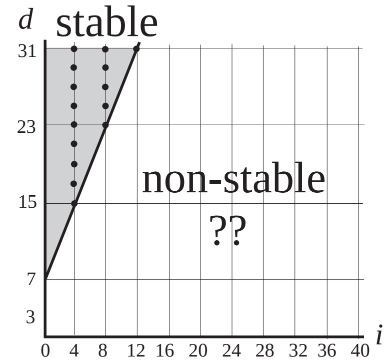
(G. Kuperberg, D. Thurston) $\dim M = 3$,

3-valent CSI $\in H^0(\sqcup_M \widetilde{B\mathrm{Diff}}(M); (\text{certain space of graphs}))$
is a universal FTI of Ohtsuki,
Goussarov-Habiro.



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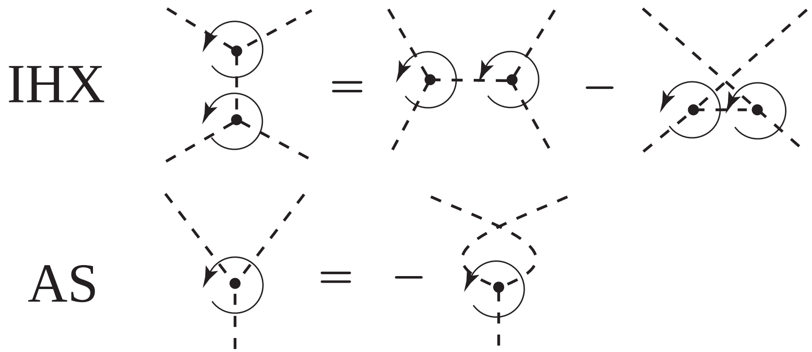
We give a higher-dim. generalization
of this to understand non-stable.

2. KONTSEVICH'S CHARACTERISTIC CLASSES

2.1. SPACE OF GRAPHS

$$\mathcal{G}_{2n} := \mathrm{span}_{\mathbf{Q}}\{\text{conn. v-ori. 3-valent graphs, } 2n\text{-vertices}\}.$$

$$\mathcal{A}_{2n} := \mathcal{G}_{2n}/\mathrm{IHX}, \mathrm{AS}.$$



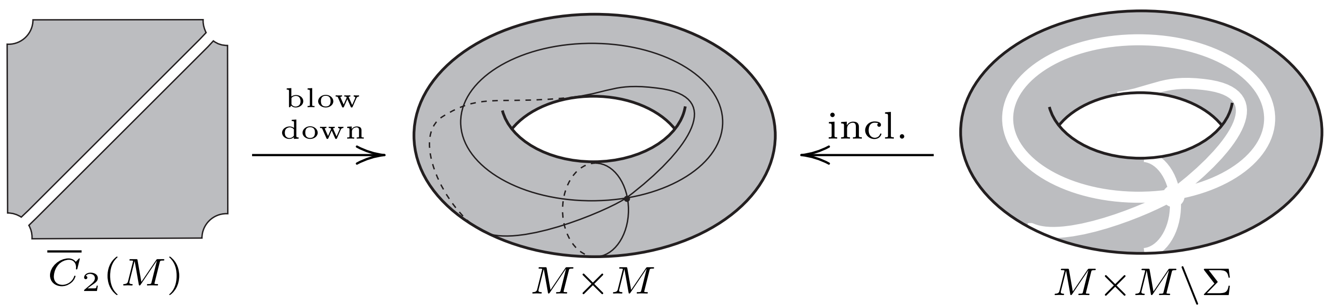
2.2. COMPACTIFICATION OF CONFIGURATION SPACE

M: (homology) $(2k+1)$-sphere with a fixed pt $\infty \in M$.

$$C_n(M \setminus \infty) := \{(x_1, \cdots, x_n) \in (M \setminus \infty)^{\times n} \,|\, x_i \neq x_j \ (i \neq j)\},$$

$$\overline{C}_n(M \setminus \infty) := \text{Fulton-MacPherson-Kontsevich compactification}$$

$$\text{of } C_n(M \setminus \infty). \quad \text{``}= B\ell_\Sigma(M^{\times n}) \text{ real blow-up''}$$



2.3. FUNDAMENTAL FORM ω ON $\overline{C}_2(M \setminus \infty)$-BUNDLE

Given a (D^{2k+1}, ∂)-bundle $\pi : E \to B$ (with $P \to B$ assoc principal),

$$\overline{C}_n(\pi) : E\overline{C}_n(\pi) \to B$$

$$E\overline{C}_n(\pi) := P \times_{\mathrm{Diff}(D^{2k+1},\partial)} \overline{C}_n(S^{2k+1} \setminus \infty)$$

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If a trivialization (framing) $\tau_E : T^{\mathrm{fib}}E \xrightarrow{\sim} \mathbf{R}^{2k+1} \times E$ given, then $\exists \omega \in \Omega^{2k}_{\mathrm{dR}}(E\overline{C}_2(\pi))$ closed form s.t.

$$\omega|_{\partial^{\mathrm{fib}} E\overline{C}_2(\pi)} = S\tau_E^* \mathrm{Vol}_{S^{2k}} \in \Omega^{2k}_{\mathrm{dR}}(\partial^{\mathrm{fib}} E\overline{C}_2(\pi)).$$

$$\begin{cases} S\tau_E : \partial^{\mathrm{fib}} E\overline{C}_2(\pi) \text{"="} S(T^{\mathrm{fib}}E) \xrightarrow{\sim} S^{2k} \times E \\ \mathrm{Vol}_{S^{2k}} : \int_{S^{2k}} \mathrm{Vol}_{S^{2k}} = 1, SO_{2k+1}\text{-invariant} \end{cases}$$

2.4. FROM GRAPHS TO DIFFERENTIAL FORMS

We define a linear map

$$\Phi : \mathcal{G}_{2n} \to \Omega^{6nk}_{\mathrm{dR}}(E\overline{C}_{2n}(\pi)) \text{ by}$$

$$\Phi(\Gamma) := \bigwedge_e \omega_e, \quad \omega_e := (E\overline{C}_{2n}(\pi) \xrightarrow{\mathrm{pr}} E\overline{C}_2(\pi))^* \omega.$$

Fiber integration $\overline{C}_{2n}(\pi)_* : \Omega^{6nk}_{\mathrm{dR}}(E\overline{C}_{2n}(\pi)) \to \Omega^{6nk-2n(2k+1)}_{\mathrm{dR}}(B)$ yields a form $\overline{C}_{2n}(\pi)_* \Phi(\Gamma) \in \Omega^{n(2k-2)}_{\mathrm{dR}}(B)$.

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Let

$$\zeta_{2n}(\pi; \tau_E) := \sum_{\Gamma} \overline{C}_{2n}(\pi)_* \Phi(\Gamma) \frac{[\Gamma]}{|\mathrm{Aut}\,\Gamma|} \in \Omega^{n(2k-2)}_{\mathrm{dR}}(B) \otimes \mathcal{A}_{2n}.$$

2.5. THEOREM (Kontsevich).

$\zeta_{2n}(\pi;\tau_E)$: *characteristic class of framed* (D^{2k+1},∂)*-bundles, i.e.,*

1. $\zeta_{2n}(\pi;\tau_E)$ *is* $(d\otimes 1)$*-closed.*
2. $[\zeta_{2n}(\pi;\tau_E)] \in H^{n(2k-2)}(B;\mathbf{R}\otimes\mathcal{A}_{2n})$ *does not depend on the closed extension* ω *chosen.*
3. $[\zeta_{2n}(\pi;\tau_E)]$ *is natural wrt maps between framed bundles.*

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3. *$[\zeta_{2n}(\pi; \tau_E)]$ is natural wrt maps between framed bundles.*

"Proof" By the generalized Stokes formula (for fiber integration) and vanishing of higher degenerations (Kontsevich's lemma),

$$(d \otimes 1)\zeta_{2n}(\pi; \tau_E) = \sum (\mathrm{IHX} + \mathrm{AS}) = 0.$$

3. FEATURES OF THE SIMPLEST CLASS

3.0. CONTENT OF THIS SECTION

- We define an unframed version $\hat{Z}_2$ of the invariant of a 'pointed' framed (D^{2k+1}, ∂)-bundle $\pi : E \to D^{2k-2}$:

$$Z_2 : \pi_{2k-2}(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)) \to \mathbf{R}$$

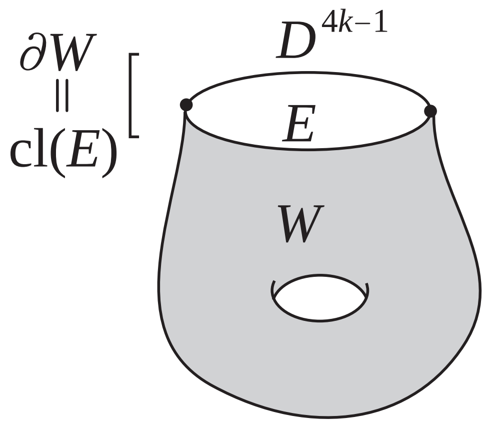
$$Z_2(\pi; \tau_E) = \zeta_2(\pi; \tau_E)[D^{2k-2}, \partial]|_{[\Theta]\mapsto 12} = \int_{E\overline{C}_2(\pi)} \omega^3 \in \mathbf{R}$$

associated to the 'Θ-graph' by introducing a correction term.

- Formula for $\hat{Z}_2 \Rightarrow \hat{Z}_2$ detects some exotic smooth structures on the total spaces.

3. FEATURES OF THE SIMPLEST CLASS

3.1. SIGNATURE DEFECT (correction term):



* $\mathrm{cl}(E) := E \cup_\partial D^{4k-1}$
  closing, canonical gluing.

* framing $\tau_E : T^{\mathrm{fib}}E \xrightarrow{\sim} \mathbf{R}^{2k+1} \times E$
  $\overset{\text{extend}}{\rightsquigarrow}$ (stable) framing τ'_E on $TW|_{\partial W=\mathrm{cl}(E)}$.

* $L_k(TW;\tau'_E)[W,\partial W]$: relative L_k-characteristic number.

* (Signature defect)

$$\Delta_k(\pi;\tau_E) := L_k(TW;\tau'_E)[W,\partial W] - \mathrm{sign}\, W$$

gives a well-defined hom. $\pi_{2k-2}(\widetilde{B\mathrm{Diff}}(D^{2k+1},\partial)) \to \mathbf{Q}$.

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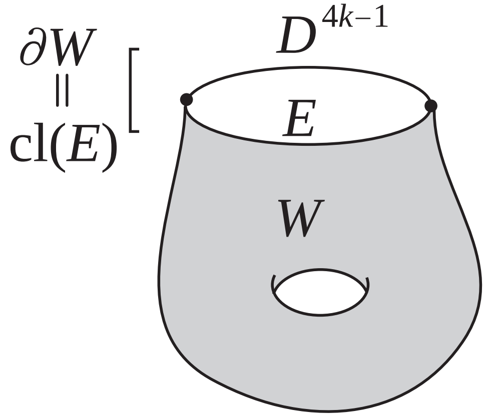

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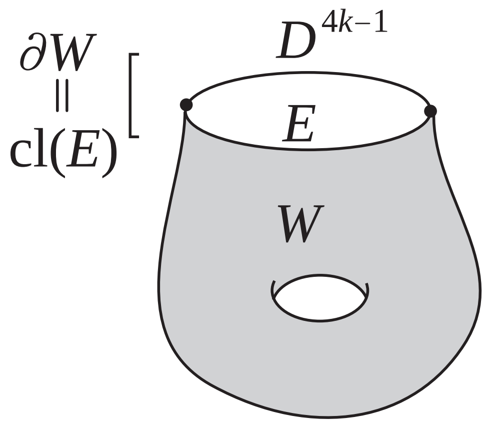

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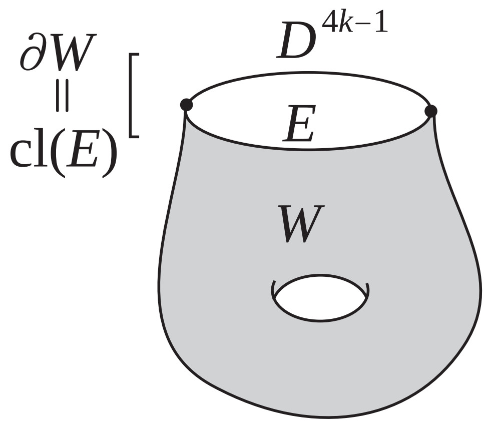

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3.2. THEOREM.

1. *The following sequence is exact.*

$$0 \to \pi_i(\Omega^d SO_d) \to \pi_i(\widetilde{B\mathrm{Diff}}(D^d,\partial)) \to \pi_i(B\mathrm{Diff}(D^d,\partial)) \to 0.$$

2. *Let $k \geq 1$. The quantity*

$$\hat{Z}_2(\pi) := Z_2(\pi;\tau_E) - \frac{(2k)!}{2^{2k+2}(2^{2k-1}-1)B_k}\Delta_k(\pi;\tau_E) \in \mathbf{Q}$$

does not depend on τ_E and thus gives a hom

$$\hat{Z}_2 : \pi_{2k-2}(B\mathrm{Diff}(D^{2k+1},\partial)) \to \mathbf{Q}.$$

3.3. IDEA OF THEOREM 3.2.

1. $\widetilde{B\mathrm{Diff}}(D^d, \partial) \simeq E\mathrm{Diff}(D^d, \partial) \times_{\mathrm{Diff}(D^d, \partial)} \Omega^d SO_d$,

$$\begin{aligned} &\mathrm{Ker}(\pi_i(\widetilde{B\mathrm{Diff}}(D^d, \partial)) \to \pi_i(B\mathrm{Diff}(D^d, \partial))) \\ &\quad = \{\text{framings on the trivial bundle}\}/\text{homotopy} \\ &\quad \cong \pi_i(\Omega^d SO_d). \end{aligned}$$

   Hence the fiber sequence splits.

2. Both Z_2 and Δ_k detect $\pi_{4k-1}(SO_{2k+1}) \otimes \mathbf{Q} = \mathbf{Q}$.
   Explicit computation relates both values.

3.4. THEOREM. *Let $k \geq 3$ and let $\pi : E \to D^{2k-2}$ be a pointed (D^{2k+1}, ∂)-bundle. If $\mathrm{cl}(E) = \partial(\mathit{Parallelizable})$, then $b_k \hat{Z}_2(\pi) \in \mathbf{Z}$ and*

$$c_k \lambda'(\mathrm{cl}(E)) \equiv b_k \hat{Z}_2(\pi) \ \mathit{mod}\ b_k, \ \ \mathit{where} \tag{1}$$

1. $b_k = 2^{2k-2}(2^{2k-1} - 1)a_k \mathrm{num}\big(B_k/k\big)$, $a_k = (3 - (-1)^k)/2$, $c_k = (2k-1)! a_k \mathrm{denom}\big(B_k/k\big)$,
2. $\lambda'(\partial W^{4k}) = \frac{\mathrm{sign}\, W^{4k}}{8}$ *mod* b_k, W^{4k}*: parallelizable, is Milnor's invariant of homotopy* $(4k-1)$*-spheres.*

REMARK (1) is a higher analogue of

$$\mathrm{Rohlin}(M^3) \equiv \mathrm{Casson}(M^3) \text{ mod } 2.$$

3.4'. THEOREM. *Let $k = 2$ and let $\pi : E \to D^2$ be a pointed (D^5, ∂)-bundle. Then $28\hat{Z}_2(\pi) \in \mathbf{Z}$ and there exists some hom $\mu : \pi_2(B\mathrm{Diff}(D^5, \partial)) \to \mathbf{Z}_2$ such that*

$$10\lambda'(\mathrm{cl}(E)) + 14\mu(\pi) \equiv 28\hat{Z}_2(\pi) \ \mathit{mod}\ 28.$$

μ is an obstruction to extending τ_E to a framing of a parallelizable manifold W^8, $\partial W^8 = \mathrm{cl}(E)$.

QUESTION (Open). Is μ non-trivial? (If so, either $\hat{Z}_2$ is non-trivial, or $\mathrm{Im}\,(\lambda' \circ \mathrm{cl}) = 7\mathbf{Z}_4 \subset \mathbf{Z}_{28}$, i.e., the Gromoll group $\Gamma^7_2 = 7\mathbf{Z}_4$.)

3.5. COROLLARY. *If $k = 4, 5$ or $12 \leq k \leq 41$, or more generally, if $k \geq 12$ and moreover*

1. *$2k-1$ is prime and*
2. *$2^{2k-1}-1$ has a prime factor p s.t.*

$$\begin{cases} p \nmid \mathrm{num}\left(\frac{B_m}{m}\right) & \text{if } k = 2m \\ p \nmid \mathrm{num}\left(\frac{B_m}{m}\right)\mathrm{num}\left(\frac{B_{m+1}}{m+1}\right) & \text{if } k = 2m+1 \end{cases}$$

(e.g. regular prime p satisfies this) then $\hat{Z}_2$ is non-trivial.

In particular, $\pi_{2k-2}(B\mathrm{Diff}(D^{2k+1}, \partial))$ is infinite for these k.

3. FEATURES OF THE SIMPLEST CLASS

3.6. IDEA OF THEOREM 3.4, 3.4' AND 3.5

- By the assumption $\mathrm{cl}(E) = \partial W$, W: parallelizable,

$$\hat{Z}_2(\pi) = Z_2(\pi;\tau_E) - \frac{1}{4}p_k(TW;\tau'_E)[W,\partial W] + \frac{c_k}{b_k}\cdot\frac{\operatorname{sign} W}{8}$$
$$= \mathbf{Z} + \left(\pm\frac{1}{2}\mu(\pi)\right) + \frac{c_k}{b_k}\cdot\lambda'(\mathrm{cl}(E)).$$

- It suffices to check $c_k\lambda'(\mathrm{cl}(E)) \not\equiv 0 \bmod b_k$ for some E.

- Use Antonelli-Burghelea-Kahn's elements(*) of $\pi_{2k-2}(B\mathrm{Diff}(D^{2k+1},\partial))$ such that the λ'-invariant are non-trivial.

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* P. L. Antonelli, D. Burghelea, P. J. Kahn, *The non-finite homotopy type of some diffeomorphism groups*, Topology **11** (1972), pp. 1–49.

4. HIGHER CLASSES

$[\zeta_{2n}] \in H^{n(2k-2)}(B; \mathbf{R} \otimes \mathcal{A}_{2n})$ for $n \geq 2$.

$\dim B = n(2k-2) \Rightarrow \langle [\zeta_{2n}], [B] \rangle \in \mathbf{R} \otimes \mathcal{A}_{2n}$ defines a hom

$\Omega_{n(2k-2)}(\widetilde{B\mathrm{Diff}}(D^{2k+1}, \partial)) \longrightarrow \mathbf{R} \otimes \mathcal{A}_{2n}$.

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4.1. THEOREM. *Let $k = 2m-1 \geq 1$ and $n \geq 2$. There is a hom*

$$\psi_{2n} : \mathcal{G}_{2n} \to \Omega_{n(4m-4)}(\widetilde{B\mathrm{Diff}}(D^{4m-1}, \partial)) \otimes \mathbf{Q}$$

s.t. $\langle [\zeta_{2n}], [\psi_{2n}(\Gamma)] \rangle \doteq [\Gamma]$ (up to a const). Furthermore, if m even,

$$\dim \pi_{n(4m-4)}(B\mathrm{Diff}(D^{4m-1}, \partial)) \otimes \mathbf{Q} \geq \dim \mathcal{A}_{2n}.$$

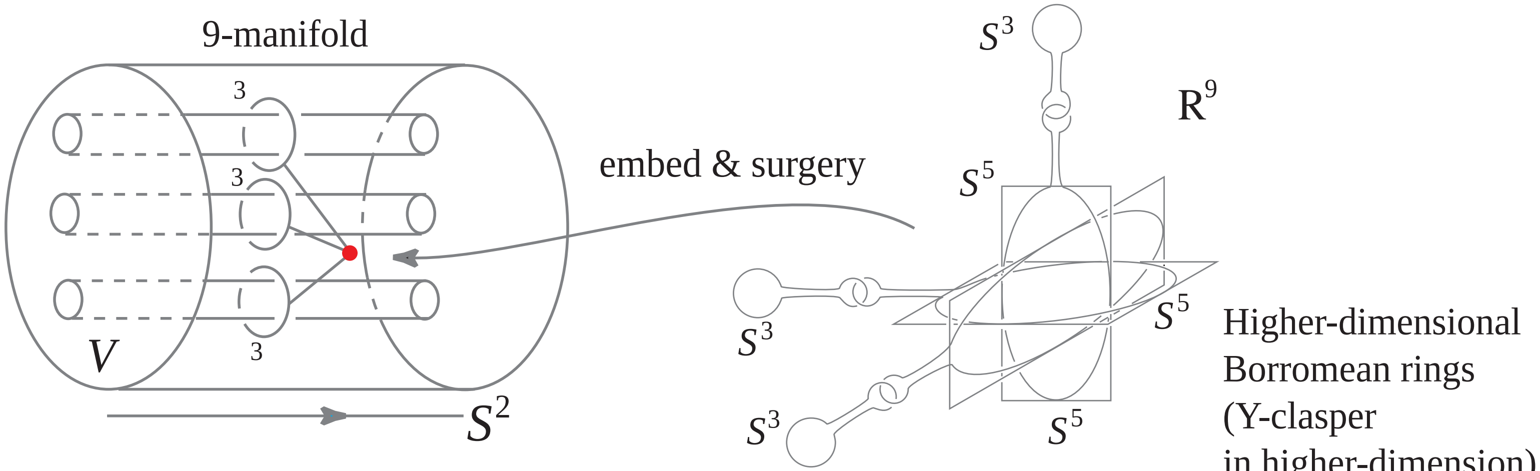
(D. Bar-Natan)

n	1	2	3	4	5	6	7	8	9	10	11
$\dim \mathcal{A}_{2n}$	1	1	1	2	2	3	4	5	6	8	9

4. HIGHER CLASSES

4.2. CONSTRUCTION (1/3) – BASIC Y-SURGERY

Assume $m = 1$ for simplicity. $(\psi_{2n} : \mathcal{G}_{2n} \to \Omega_{4n}(\widetilde{B\mathrm{Diff}}(D^7, \partial)) \otimes \mathbf{Q})$
$V = \overline{\text{reg-nh}}(S^3 \vee S^3 \vee S^3) \subset \mathrm{Int}(D^7)$.



LEMMA $\exists$ *framed* (V, ∂)*-bundle structure* $\pi^{Y^{\cup r}} : (V \times S^2)^{Y^{\cup r}} \to S^2$ *(Y-surgery r times) for* $\exists r \in \mathbf{Z}$ *extending the trivial* ∂V*-bundle.*

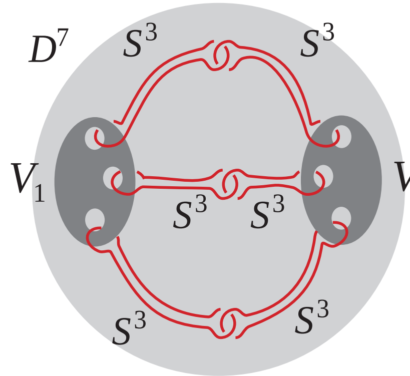
4.3. CONSTRUCTION (2/3) – Y-SURGERY ON A BUNDLE

- Given a (D^7, ∂)-bundle $\pi : E \to B$, we replace a trivial subbundle $V \times B \subset E$ with the pullback (V, ∂)-bundle $f^*(V \times S^2)^{Y^{\cup r}}$ by a map $f : B \to S^2$.

- We denote the resulting (framed) bundle by

$$\pi^{Y^r(V \times B, f)} : E^{Y^r(V \times B, f)} \to B.$$

4.4. CONSTRUCTION (3/3) – Γ-SURGERY



* Embed $2n$ handlebodies $\cong V$ in a single D^7 disjointly "along Γ", with some framed links.
* Direct product with $(S^2)^{\times 2n}$ includes $2n$ trivial sub V-bundles $\widehat{V}_1, \ldots, \widehat{V}_{2n}$.
* Let $\psi_{2n}(\Gamma) \in \Omega_{4n}(\widetilde{B\mathrm{Diff}}(D^7, \partial))$ be given by the bundle

$$\pi^\Gamma := \pi^{Y^r(\prod_i \widehat{V}_i, \prod_i f_i) \cup \text{links}} : (D^7 \times (S^2)^{\times 2n})^{Y^r(\prod_i \widehat{V}_i, \prod_i f_i) \cup \text{links}} \to (S^2)^{\times 2n}$$

where $f_i : (S^2)^{\times 2n} \to S^2$, $(x_1, \ldots, x_{2n}) \mapsto x_i$.

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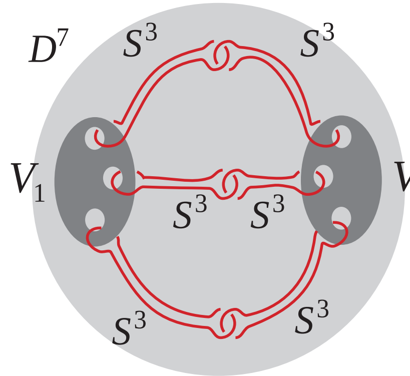

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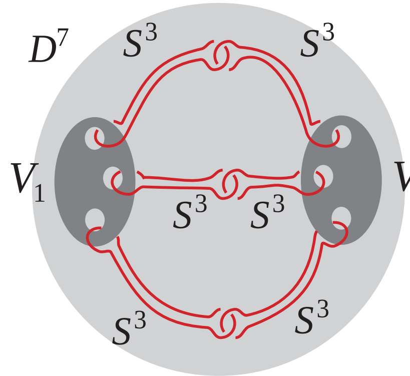

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4.5. EVALUATION (1/2)

Given $V_i \sqcup V_j \subset D^7$,
we say that a configuration $(x, y) \in \overline{C}_2(D^7 \setminus \partial)$ is *separated* if $x \in V_i, y \in V_j$ for some $i \neq j$.

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LEMMA (Lescop's lemma in higher-dim)
$\exists \omega \in \Omega^6_{\mathrm{dR}}(E\overline{C}_2(\pi^\Gamma))$ *s.t.*

1. $\omega|_{E\overline{C}_2(\pi^\Gamma)\setminus(\text{separated})}$ *is a pullback of a form on less dimensions.*
2. $\exists A^{(i)}_k(t)_{t \in S^2} \in \Omega^3_{\mathrm{dR}}((V_i \times S^2)^{Y^r})$,
   $A^{(j)}_k(t)_{t \in S^2} \in \Omega^3_{\mathrm{dR}}((V_j \times S^2)^{Y^r})$ *s.t.*
$$\omega|_{(V_i \times V_j)_{(t_1, t_2) \in S^2 \times S^2}} = \sum_k f_1^* A^{(i)}_k(t_1) \wedge f_2^* A^{(j)}_k(t_2).$$

4.6. EVALUATION (2/2) Hence, only mutually separated configs contribute to the integral:

$$\int_{(S^2)^{\times 2n}} \overline{C}_{2n}(\pi^{\Gamma})_* \Phi(\Gamma') = \int_{(S^2)^{\times 2n}} \sum_{\sigma \in \mathfrak{S}_{2n}} \int^{\mathrm{fib}}_{V_{\sigma(1)} \times \cdots \times V_{\sigma(2n)}} \Phi(\Gamma'),$$

$$\begin{aligned}
\langle [\zeta_{2n}], [\psi_{2n}(\Gamma)] \rangle &= \sum_{\Gamma'} \frac{[\Gamma']}{|\mathrm{Aut}\,\Gamma'|} \int_{(S^2)^{\times 2n}} \sum_{\sigma \in \mathfrak{S}_{2n}} \int^{\mathrm{fib}}_{V_{\sigma(1)} \times \cdots \times V_{\sigma(2n)}} \Phi(\Gamma') \\
&\doteq [\Gamma] \int_{\substack{(t_1, \ldots, t_{2n}) \\ \in (S^2)^{\times 2n}}} \int^{\mathrm{fib}}_{V_1 \times \cdots \times V_{2n}} B^{(1)}(t_1) \wedge \cdots \wedge B^{(2n)}(t_{2n}) \\
&= [\Gamma] \prod_{i=1}^{2n} \int_{t_i \in S^2} \int^{\mathrm{fib}}_{V_i} B^{(i)}(t_i) \doteq [\Gamma].
\end{aligned}$$

5. REMARK

QUESTION 1. Is $\pi_i(B\mathrm{Diff}(D^d, \partial))$ finitely generated?

2. Does $\mathrm{Diff}(D^d, \partial)$ has the homotopy type of a finite CW complex? for $d = 4, 5, 6$. If so, $\mathrm{Diff}(D^d, \partial) \simeq S^1 \times \cdots \times S^1$.
   - (Hatcher) $\mathrm{Diff}(D^3, \partial) \simeq *$.
   - (Antonelli-Burghelea-Kahn) $\mathrm{Diff}(D^d, \partial)$ for $d \geq 7$ is non-finite.