# LAMBDA LENGTHS

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ABSTRACT. These are notes derived from a master's class on Decorated Teichmüller Theory taught at Aarhus University during August, 2006, under the aegis of the Center for the Topology and Quantization of Moduli Spaces. The current working manuscript covers only the first half of these lectures, and it is planned to complete this manuscript with an exposition of the second half as well.

## INTRODUCTION

These notes are intended to be a self-contained and elementary presentation from first principles of Decorated Teichmüller Theory, as developed by the author and collaborators over the last 20 years, with an eye towards presenting the geometric background necessary for the quantization of Teichmüller space [13, 35] and for cluster algebras [16, 17]. There are also digressions into other applications, for instance, to algebraic number theory, circle homeomorphisms, harmonic analysis, the topology of Riemann's moduli space and arc complexes, Morita-Johnson theory, the punctured solenoid, and string theory.

We have taken this opportunity to correct a few small calculational errors (which are explicitly noted in the text) and to present sometimes simpler and sometimes more detailed proofs than in the original papers. Though the theory was originally developed [53, 54] for punctured surfaces, there are also analogous results for surfaces with boundary [62], for "partially decorated surfaces", and for "surfaces with holes". We have furthermore taken this opportunity to develop these other parallel versions here in part because of their relevance to cluster algebras and possible relevance to quantization.

These notes are organized as follows. The first lecture introduces the objects of central interest: the action of the mapping class group

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on Teichmüller space with quotient Riemann's moduli space. The upper half-plane, Poincaré disk, and Minkowski models for the hyperbolic plane are described in the second lecture together with the isometric actions of the Möbius group on these spaces and several subsequently useful formulas. The third lecture digresses from geometry to discuss algebraic number theory and in particular gives a geometric description of the Gauss product of definite binary quadratic forms; in fact, one of the author's papers [59] (©Springer-Verlag 1996) on this material, which was again intended to be a self-contained introduction from first principles, is included as an appendix with the kind permission of Springer Science and Business Media. The fourth lecture introduces our "lambda length" coordinates, which are the basis for essentially all of our constructions and calculations, and several fundamental formulas are derived via direct calculation in Minkowski space. The fifth lecture finally applies this material to give global coordinates on the decorated Teichmüller space of a punctured surface and to prove a number of basic algebraic and geometric properties. The sixth lecture describes the parametrization of the Teichmüller space of a surface with holes using "shear coordinates" (which are basic to Y-systems in cluster algebras), explains the combinatorial method for calculating holonomies (which is required for quantization), and again derives several basic algebraic and geometric properties. The seventh lecture applies the lambda lengths to study the group of homeomorphisms of the circle, and the eighth explains the corresponding Lie algebra structure together with applications to harmonic analysis and signal processing.

These notes cover somewhat more than was treated in the lectures, and though a sequential reading of the lectures is desirable, it is therefore possible to simply skip certain sections entirely without affecting continuity. The latter part of the third lecture on Gauss product can be skipped without affecting later material, and on the other hand, the earlier material in the third lecture on the Farey tesselation is required in the seventh and eighth lectures. The sixth lecture can be skipped without loss of continuity for the reader interested only in punctured surfaces, and the seventh and eighth can be skipped by the reader interested specifically in geometry.

We have not strived for completeness in the bibliography, rather, we have cited papers and books whose bibliographies should be consulted for more complete references. Let us apologize here and now if the concomitant omissions from our listed references might cause offense.

**Acknowledgments** It is a great pleasure to first and foremost thank Joergen Ellegaard-Andersen for organizing the master's class, for his input on these notes, and for arranging the visit to Aarhus. Thanks are likewise due to the other participants in the class, who made many valuable and sometimes critical comments, including Marcel Bökstedt, Bill Browder, Magnus Lauridsen, especially Gregor Masbaum, Rasmus Villemose, and Yannis Vlassopoulos. Further thanks are due to Volodya Fock, Rinat Kashaev, and Dylan Thurston for useful comments and discussions.

## 1. Cast of characters

Let $F = F_{g,r}^s$ be a smooth oriented surface of genus $g \geq 0$ with $s \geq 0$ punctures and $r \geq 0$ smooth boundary components. We shall usually require that $2g - 2 + r + s > 0$, i.e., $F$ has negative Euler characteristic, and that $r + s \geq 0$, and we shall often write $F_g^s = F_{g,0}^s$ for notational simplicity. Define the *mapping class group* $MC(F)$ of $F$ to be the group of all isotopy classes of orientation-preserving homeomorphisms of $F$, where the homeomorphisms and isotopies are required to fix each point of the boundary of $F$, and define the *pure mapping class group* $PMC(F) < MC(F)$ to be the subgroup corresponding to homeomorphisms which furthermore fix each puncture.

Really, these notes are dedicated just to the study of these groups which play a central role in low-dimensional topology and geometry. See [6] for an extensive discussion of these and related groups. However, our analysis of these groups will take us far afield from the purely group-theoretic considerations with which we begin in this lecture.

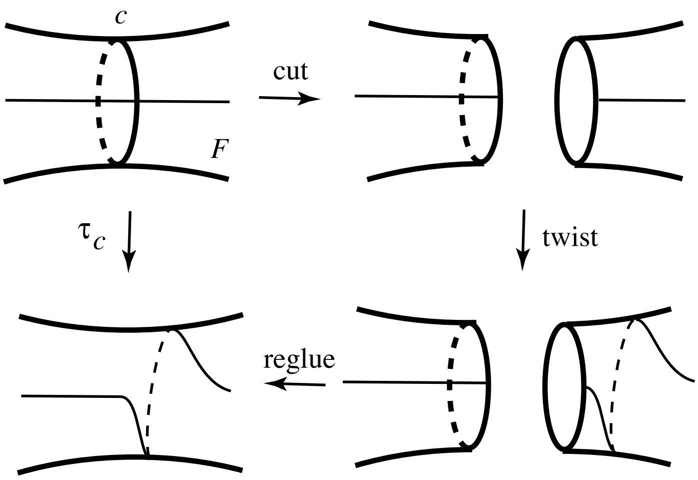



**Figure 1** Dehn twist homeomorphisms.

There are especially simple and explicit generators for $MC(F)$ going back to Max Dehn in the 1930's as follows. Suppose that $c$ is a simple closed curve in $F$ and define the *(right) Dehn twist* $\tau_c : F \to F$ to be the homeomorphism supported on an annular neighborhood of $c$, where one cuts $F$ along $c$, twists once to the right in the annular neighborhood, and then re-glues along $c$ as indicated in Figure 1, where we illustrate the affect of $\tau_c$ on an arc transverse to $c$. Notice that the isotopy class of $\tau_c$ is independent of the choice of annular neighborhood and depends only on the isotopy class of $c$ in $F$, and that the right-handed sense of $\tau_c$ depends only upon the orientation of $F$ and not upon any orientation of $c$. Furthermore, $\tau_c$ is isotopic to the identity if and only if $c$ is *inessential*, i.e., contractible to a point in $F$ or to one of the punctures of $F$.

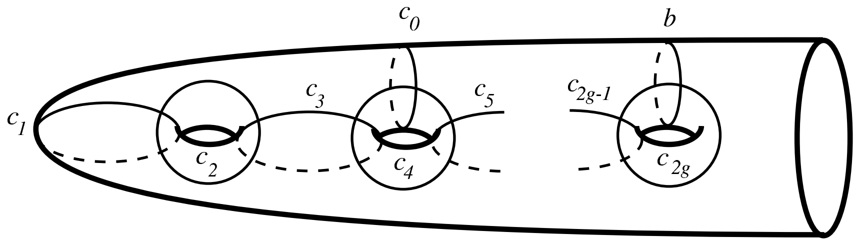



**Figure 2** Dehn twist generators for the mapping class group.

**Theorem 1.1.** [Dehn, Lickorish, Humphreys] [24] *Adopting the notation of Figure 2 for the curves* $c_0, \ldots, c_{2g}$ *in the surface* $F^0_{g,1} \subset F^0_g$*, the mapping class groups* $MC(F^0_{g,1})$ *and* $MC(F^0_g)$ *are generated by the Dehn twists* $\tau_{c_0}, \ldots, \tau_{c_{2g}}$.

Let us remark that analogous generators are known for surfaces with $s > 0$ and $r \geq 1$ as well, and also for the pure mapping class groups (cf. [6]), but we will be satisfied here with this result. Furthermore, a different generating set will be derived for any surface with $r + s \neq 0$ later in these notes (cf. Lecture 13).

In fact, there are standard relations among these generators that were essentially already known to Dehn as follows, where we assume that all curves mentioned are simple closed curves.

(**Naturality**) if $f \in MC(F)$ with $f(c) = d$, then $\tau_d = f\tau_c f^{-1}$;

(**Commutativity**) if $c$ and $d$ are disjoint, then $\tau_c\tau_d = \tau_d\tau_c$;

(**Braiding**) if $c$ and $d$ meet transversely in a single point, then $\tau_c\tau_d\tau_c = \tau_d\tau_c\tau_d$;

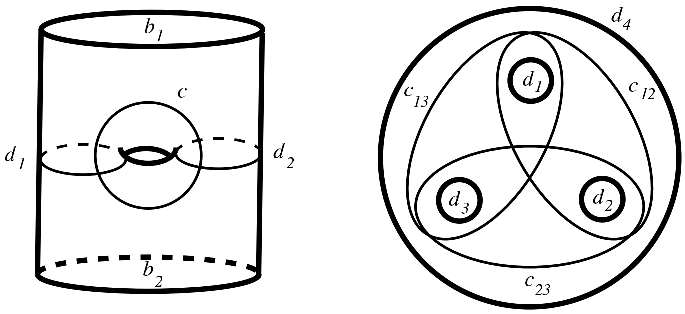
(**Chain**) if $d_1, d_2$ are disjoint each meeting $c$ transversely in a single point as in Figure 3a, then $(\tau_{d_1}\tau_c\tau_{d_2})^4 = \tau_{b_1}^{-1}\tau_{b_2}^{-1}$, where $b_1, b_2$ are the boundary components of a neighborhood in $F$ of $c \cup d_1 \cup d_2$;

(**Lantern**) if $c_{12}, c_{23}, c_{13}$ is a configuration of curves in $F_{0,4}^0$ pairwise intersecting one another transversely in pairs of points, where $d_1, d_2, d_3, d_4$ denote the boundary components of $F_{0,4}^0$ as in Figure 3b, then $\tau_{c_{12}}\tau_{c_{23}}\tau_{c_{13}} = \tau_{d_1}\tau_{d_2}\tau_{d_3}\tau_{d_4}$;

(**Garside**) in the notation of Figure 2, if we define the *Garside word*

$$\Delta_g = (\tau_{c_1})(\tau_{c_2}\tau_{c_1})(\tau_{c_3}\tau_{c_2}\tau_{c_1})\cdots(\tau_{c_{2g}}\cdots\tau_{c_2}\tau_{c_1}),$$

then $\Delta_g^4$ is the Dehn twist along the boundary of $F_{g,1}^0 \subset F_g^0$.




**3a** Chain relation **3b** Lantern relation

**Figure 3** The chain and lantern relations.

The naturality and commutativity relations follow directly from the definitions. The remaining relations are actually kind of fun to verify, where one chooses a collection of arcs in each case that decomposes the surface in question into a disk, one confirms that the two sides of the asserted equation maps each such arc to pairs of isotopic arcs, and one finally uses Alexander's trick (that a homeomorphism of a disk which is the identity on the boundary is isotopic to the identity) to conclude that indeed the asserted relations hold in $MC(F)$. These verifications give the flavor of the combinatorial fun to be had with this description of the mapping class group.

Let us say that a relation as above is "non-separating" in $F$ if all of the curves in the relation are non-separating in the surface $F$. It required a long sequence of ideas from Dehn to Cerf to Hatcher/Thurston to Wajnryb to show that this list of easily verified relations in fact give a complete presentation of $MC(F)$:

**Theorem 1.2.** [Wajnryb] [7, 69] *Take the generators for $MC(F^0_{g,1})$ described in Theorem 1.1, omitting $c_0$ for $g = 1$. Then a complete set of relations for $MC(F^0_{g,1})$ are provided by commutativity and braiding among these generators plus the following relations: for $g = 1$, no additional relations; for $g = 2$, a single non-separating chain relation; for $g \geq 3$, a single non-separating chain relation and a single non-separating lantern relation. Furthermore, the kernel of the natural homomorphism $MC(F^0_{g,1}) \to MC(F^0_g)$ is generated by the following relations: for $g = 1$, $\Delta_1^4 = 1$, and for $g \geq 2$, $\Delta_g^2 \tau_b \Delta_g^{-2} = \tau_b$, where $b$ is the curve illustrated in Figure 2.*

**Corollary 1.3.** *For $g \geq 3$, we have $H_1(MC(F^0_g)) = 0$, i.e., the abelianization of the mapping class group is trivial.*

*Proof.* According to Theorem 1.1, the mapping class group of $F = F^0_g$ is generated by Dehn twists along non-separating curves. If $c$ and $d$ are non-separating curves, then $F - c$ is homeomorphic to $F - d$, and so $\tau_c$ is conjugate in $MC(F)$ to $\tau_d$ by the naturality relation. It follows that the abelianization of $MC(F)$ is a cyclic group. Since there is a non-separating lantern relation in $F$ for $g \geq 3$ and since the exponent sum of a lantern relation is one, it follows that this cyclic group is trivial. □

Further analogous tricks with the lantern relation prove that $MC(F^s_{g,r})$ also abelianizes to zero for $g \geq 3$. This elementary argument belies the difficulty of the following important and fundamental open problem:

**Problem 1.4.** Calculate the homology or cohomology of the mapping class groups.

We shall develop tools towards this problem by considering actions of mapping class groups on appropriate spaces as follows. Supposing now that $r = 0$ and $2g - 2 + s > 0$, first roughly define the "Teichmüller space" $T(F)$ of $F = F^s_g$ in any one of the following equivalent ways:

$$
\begin{aligned}
T(F) &= \{\text{complex structures on } F\}/\text{isotopy} \\
&= \{\text{conformal structures on } F\}/\text{isotopy} \\
&= \{\text{hyperbolic structures on } F\}/\text{isotopy}.
\end{aligned}
$$

Slightly more explicitly in the first two formulations, we consider complex or conformal structures on the fixed smooth manifold $F$ modulo push-forward of structure by diffeomorphisms isotopic to the identity, where the equivalence of conformal and complex structures in this dimension is easily verified. See [1, 22] for a more complete discussion.

In fact, it is the third formulation we shall develop here, where by a "hyperbolic structure" on $F$ we mean a complete finite-area Riemannian metric of constant Gauss curvature -1 again modulo push-forward by diffeomorphisms isotopic to the identity. The equivalence of conformal and hyperbolic structures is provided by the celebrated Uniformization Theorem of Koebe, Klein, and Poincaré. See [1, 22] for more details.

Let us now be precise with a careful definition of $T(F)$ as a topological space in the hyperbolic setting. Define the *Möbius group* $PSL_2(\mathbb{R})$ to be the quotient of the group of two-by-two matrices of determinant one over the reals $\mathbb{R}$, where the matrix

$$
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is identified with } -A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}.
$$

(Much more will be said about this group in the next several lectures, and see [5, 18] for further details.) Consider the collection of homomorphisms $\rho : \pi_1(F) \to PSL_2(\mathbb{R})$ from the fundamental group $\pi_1(F)$ to $PSL_2(\mathbb{R})$, and define the collection $Hom'(\pi_1(F), PSL_2(\mathbb{R}))$ of all such homomorphisms $\rho$ so that $\rho$ is injective ($\rho$ is *faithful*), the identity in $PSL_2(\mathbb{R})$ is not an accumulation point of the image of $\rho$ ($\rho$ is *discrete*), and finally if $\gamma \in \pi_1(F)$ is freely homotopic to a puncture of $F$, then the absolute value of the trace of $\rho(\gamma)$ is 2 ($\rho$ *maps peripheral elements to parabolics*). Choosing a basis for $\pi_1(F)$, this space of homomorphisms inherits a topology from that on the entries of the representing matrices; this topology is independent of the choice of basis since given two such choices, it is easy to see that one topology is finer than the other. Finally, $PSL_2(\mathbb{R})$ acts on this space of homomorphisms by conjugation, and we define the quotient space

$$
T(F) = Hom'(\pi_1(F), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}),
$$

to be the *Teichmüller space* of $F$.

This may seem like a mouthful, but we shall develop tools to understand this space quite explicitly from geometric and combinatorial points of view. In fact, we shall see (cf. Theorems 5.2 and 6.1) that $T(F)$ is homeomorphic to an open ball of dimension $6g-6+2s$ for $F = F_g^s$ with $s \geq 1$ (this result holding also for $s = 0$). The principal reason for introducing $T(F)$ is that $MC(F)$ acts on this ball, as is obvious from the rough definitions since in each case we take the quotient by push-forward of structure by isotopy just as in the definition of the mapping class group. In our careful definition, $MC(F)$ acts on $\pi_1(F)$ by outer automorphisms, so the action on $T(F)$ is well-defined since we mod out by conjugation. Furthermore, we shall see that $MC(F)$ acts on $T(F)$ discretely but with fixed points of finite isotropy (i.e., the stabilizer in $MC(F)$ of any point of $T(F)$ is a finite subgroup), and these properties allow us to derive facts about $MC(F)$ from this action. The quotient

$$M(F) = T(F)/MC(F)$$

is *Riemann's moduli space* and is arguably the other central character of our considerations beyond the mapping class group. Closely related to Problem 1.4, we have the likewise fundamental open problem:

**Problem 1.5.** Calculate the homology or cohomology groups of Riemann's moduli spaces.

In fact, we shall almost always require $s \geq 1$ in the sequel and work with the trivial $\mathbb{R}^s_{>0}$-bundle over $T(F)$, where $F = F_g^s$, which we call the *decorated Teichmüller space* and denote $\widetilde{T}(F)$. At the moment, this is simply an abstract bundle over $T(F)$, but we shall develop a natural geometric interpretation for it as well. Thinking of a point of $\widetilde{T}(F)$ as a point in $T(F)$ together with the assignment of a positive real number to each puncture, the mapping class group $MC(F)$ also acts on $\tilde{T}(F)$ by simply permuting the assigned numbers. Thus, the basic cast of characters for these notes is summarized in the following diagram:

$$\begin{array}{rl}
MC(F) \circlearrowright & \widetilde{T}(F) - \mathbb{R}^s_{>0} \\
& \downarrow \\
MC(F) \circlearrowright & T(F) \\
& \downarrow \\
& M(F)
\end{array}$$

## 2. Three models for the hyperbolic plane

The discovery of two-dimensional hyperbolic geometry in the nineteenth century independently by Klein, Poincaré and Lobochevsky solved a millennia-old problem on the consistency and independence of Euclid's other axioms from his Fifth Axiom: given a line and a point not on that line, there is a unique line through the given point which is parallel to the given line. This lecture introduces three different models for hyperbolic geometry which are basic to our treatment of Teichmüller space. References for the material in this lecture are [5, 18, 53]

The first model is the *upper half-plane*

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

endowed with the Riemannian metric $ds^2 = (dx^2 + dy^2)/y^2$, so $\mathcal{U}$ is conformal to the usual Euclidean upper half-plane. We shall typically identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and hence $\mathcal{U} \subset \mathbb{C}$ with the complex numbers $z = x + iy$ of positive imaginary part $y > 0$, where $i = \sqrt{-1}$.

One can check without difficulty that geodesics in $\mathcal{U}$ are given by circles perpendicular to the real axis together with the extreme case of rays with real endpoints parallel to the imaginary axis. Thus, one sees immediately that Euclid's Fifth Axiom fails since there are in fact infinitely many lines parallel to a given line passing through a given point not on the given line, and it is an exhaustive exercise to check that Euclid's remaining axioms do indeed hold in the upper half-plane. A further calculation shows that $\mathcal{U}$ has constant Gauss curvature $-1$. Let us emphasize that points of the extended real axis $\mathbb{R}\cup\{\infty\}$ are *not* points in $\mathcal{U}$, but rather comprise a "circle at infinity" compactifying $\mathcal{U}$ to a closed ball.

The Möbius group $PSL_2(\mathbb{R})$ was already defined in the previous lecture, and an element of this group, which is called a *Möbius transformation*, acts on $\mathcal{U}$ by *fractional linear transformation*:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

One directly checks that $\mathcal{U}$ is preserved by each Möbius transformation, that this is in fact an action by orientation-preserving isometries of $\mathcal{U}$, and that $PSL_2(\mathbb{R})$ is indeed the full group of orientation-preserving isometries of $\mathcal{U}$, where the last assertion follows from the easily verified fact that the Möbius group acts transitively on the tangent space to $\mathcal{U}$.

One reason that the upper half-plane is useful for calculations in hyperbolic geometry is this simple expression for the action of isometries.

Since the fixed points $z = (az+b)/(cz+d)$ evidently satisfy the equation $2cz = a-d \pm \sqrt{(a+d)^2-4}$ for $c \neq 0$ (and for $c=0$, we may conjugate by an appropriate Möbius transformation $B$ noting that the fixed points of $BAB^{-1}$ are the images of the fixed points of $A$ under $B$), direct calculation shows that there is the following trichotomy on $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{R})$:

• if $|a+d| < 2$, then $A$ is said to be *elliptic*, and there is unique fixed point in $\mathcal{U}$ with $A$ described as rotation about this fixed point;

• if $|a+d| = 2$, then $A$ is said to be *parabolic*, and there is a unique fixed point at infinity;

• if $|a+d| > 2$, then $A$ is said to be *hyperbolic*, and there is a pair of fixed points at infinity, which determine a unique geodesic in $\mathcal{U}$ with $A$ acting as translation along this geodesic. Furthermore, the translation length $\ell_A$ along this invariant geodesic is given by $2\cosh \frac{\ell_A}{2} = |a+d|$.

The elliptic and hyperbolic cases are analogues of the familiar rotations and translations in Euclidean geometry, but the parabolic case is a novel aspect compared to the familiar case. (We shall comment further on parabolic transformations later in this lecture.)

Our second model for the hyperbolic plane is the *Poincaré disk*

$$\mathbb{D} = \{z \in \mathbb{C} : |z| = x^2 + y^2 < 1\}$$

endowed with the metric $ds^2 = 4(dx^2+dy^2)/(1-|z|^2)^2$, so this is again a model conformal to the Euclidean metric on the open unit disk in $\mathbb{C}$. Whereas the upper half-plane is useful for calculations, the Poincaré disk is useful for drawing pictures in part because the point $\infty$ at infinity in $\mathcal{U}$ is seen to be "just another point" in the unit circle $S^1_\infty$ in $\mathbb{C}$, the *circle at infinity*, which compactifies $\mathbb{D}$ to the closed unit disk in $\mathbb{C}$.

There are explicit inverse isometries, called the *Cayley transform*, between $\mathcal{U}$ and $\mathbb{D}$ as follows:

$$\begin{array}{cc} \mathcal{U} \to \mathbb{D}, & \mathbb{D} \to \mathcal{U}. \\ z \mapsto \dfrac{z-i}{z+i} & w \mapsto i\,\dfrac{1+w}{1-w} \end{array}$$

which map the respective points $0, 1, \infty$ at infinity in $\mathcal{U}$ to the points $-1, -i, +1 \in S^1_\infty$. The action of the Möbius group on $\mathbb{D}$ is given by conjugating the action of fractional linear transformations on $\mathcal{U}$ by the Cayley transform.

Our third model for the hyperbolic plane is useful both for figures and for calculations as we shall see. It is described as a subset of *Minkowski 3-space*, which is defined to be $\mathbb{R}^3$ endowed with the indefinite pairing

$$\begin{aligned} < \cdot, \cdot > : \mathbb{R}^3 \times \mathbb{R}^3 &\rightarrow \mathbb{R} \\ < (x, y, z), (x', y', z') > &\mapsto xx' + yy' - zz', \end{aligned}$$

where $x, y, z$ denote the usual coordinates on $\mathbb{R}^3$.

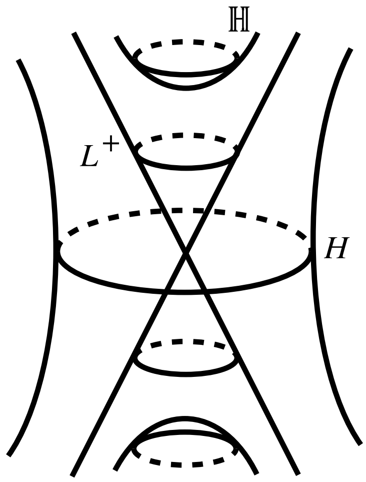



**Figure 4** Minkowski 3-space

There are several characteristic subspaces of Minkowski 3-space, which are illustrated in Figure 4, as follows:

the *upper sheet of the hyperboloid*

$$\mathbb{H} = \{u = (x, y, z) \in \mathbb{R}^3 : < u, u >= -1 \text{ and } z > 0\};$$

the *open positive light-cone*

$$L^+ = \{u = (x, y, z) \in \mathbb{R}^3 : < u, u >= 0 \text{ and } z > 0\};$$

and the *hyperboloid of one sheet*

$$H = \{u = (x, y, z) \in \mathbb{R}^3 : < u, u >= +1\}.$$

We shall say that an affine plane $\Pi$ in $\mathbb{R}^3$ is *elliptic, parabolic, or hyperbolic*, respectively, if it is so in the sense of the Greeks, i.e., the corresponding conic section $\Pi \cap (L^+ \cup -L^+)$ has the corresponding attribute. One checks without difficulty that a plane

$$\Pi = \{u \in \mathbb{R}^3 : < u, v >= \xi\}$$

with Minkowski normal $v$, for some $\xi \in \mathbb{R}$, is elliptic, parabolic, or hyperbolic, respectively, if and only if $< v, v >$ is negative, vanishes, or is positive. One furthermore easily proves:

**Lemma 2.1.** *The Minkowski pairing $< \cdot, \cdot >$ restricts to a bilinear pairing on $\Pi$ that is definite, degenerate, or indefinite, respectively, precisely when $\Pi$ is elliptic, parabolic, or hyperbolic.*

In particular, the upper sheet $\mathbb{H}$ of the hyperboloid has all tangent planes elliptic and so inherits an honest Riemannian metric from the indefinite pairing on Minkowski space. The upper sheet $\mathbb{H}$ together with this induced metric forms our third model for the hyperbolic plane.

Identify the closed unit disk $\mathbb{D} \cup S^1_\infty$ with the horizontal disk at height zero in Minkowski space. An isometry between $\mathbb{H}$ and $\mathbb{D}$ is given by central projection $\bar{\cdot}$ from the point $(0, 0, -1)$, i.e., the line segment with endpoints $(0, 0, -1)$ and $v \in \mathbb{H}$ meets the unit disk $\mathbb{D}$ at the point $\bar{v}$; that is,

$$\begin{aligned} \bar{\cdot} : \mathbb{H} &\to \mathbb{D} \\ (x, y, z) &\mapsto \overline{(x, y, z)} = \frac{1}{1+z}(x, y) \end{aligned}$$

establishes an isometry between these two models, where the inverse is given by

$$\begin{aligned} \mathbb{D} &\to \mathbb{H} \\ (x, y) &\mapsto \frac{1}{1 - x^2 - y^2}(2x, 2y, 1 + x^2 + y^2). \end{aligned}$$

Furthermore, this projection extends to a natural mapping

$$\begin{aligned} \bar{\cdot} : L^+ &\to S^1_\infty \\ (x, y, z) &\mapsto \overline{(x, y, z)} = \frac{1}{x^2 + y^2}(x, y, 0), \end{aligned}$$

and the fiber over a point of $S^1_\infty$ is a corresponding ray in $L^+$.

Finally composing this central projection with the Cayley transform, explicit formulas that will be useful in subsequent calculations give isometries:

$$\begin{aligned} \mathbb{H} &\to \mathcal{U} \\ (x, y, z) &\mapsto (-y + i)/(z - x) \end{aligned}$$

and

$$\begin{aligned} \mathcal{U} &\to \mathbb{H} \\ x+iy &\mapsto \frac{1}{2y}\ (x^2+y^2-1, -2x, x^2+y^2+1) \end{aligned}$$

The action of $PSL_2(\mathbb{R})$ on $\mathbb{H}$ can be computed by conjugating its action on $\mathcal{U}$ by these isometries, and one finds that $PSL_2(\mathbb{R})$ is exactly the group of real three-by-three matrices of determinant one which preserve the Minkowski pairing as well as preserving the upper sheet $\mathbb{H}$, i.e., $PSL_2(\mathbb{R})$ is isomorphic to the component $SO^+(2,1)$ of the identity in $SO(2,1)$, which is of index four (preserving not only the orientation of $\mathbb{R}^3$ but also the upper sheet $\mathbb{H}$ of the hyperboloid).

More explicitly, we may identify $\mathbb{R}^3$ with the collection $\mathcal{B}_2(\mathbb{R})$ of all real symmetric bilinear forms on two indeterminates via

$$\begin{aligned} \mathbb{R}^3 &\to \mathcal{B}_2(\mathbb{R}) \\ (x,y,z) &\mapsto \begin{pmatrix} z-x & y \\ y & z+x \end{pmatrix}, \end{aligned}$$

and then the action of $A \in PSL_2(\mathbb{R})$ is simply given by

$$\begin{aligned} A : \mathcal{B}_2(\mathbb{R}) &\to \mathcal{B}_2(\mathbb{R}) \\ B &\mapsto A^tBA, \end{aligned}$$

where $A^t$ denotes the transpose of $A$. Thus, the natural action of the Möbius group by change of basis for bilinear forms coincides with its action as hyperbolic isometries. (This theme will be further developed in the next lecture.)

Since an isometry of Minkowski space preserving an affine plane must preserve its normal vectors, we conclude:

**Lemma 2.2.** *If a Möbius transformation $A$ leaves invariant an affine plane which is elliptic, parabolic, or hyperbolic, respectively, then $A$ itself must also be elliptic, parabolic, or hyperbolic.*

Geodesics in $\mathbb{H}$ are intersections with planes through the origin whose Minkowski normal lies in $H$ as one can check directly, and several standard formulas relating Minkowski inner products and hyperbolic lengths and angles can then also be directly derived:

**Lemma 2.3.** *If $u, v \in \mathbb{H}$, then*

$$< u, v >^2 = \cosh^2\delta,$$

*where $\delta$ is the hyperbolic distance between $u$ and $v$. If $u \in \mathbb{H}$ and $v \in H$, then*

$$< u, v >^2 = \sinh^2\delta,$$

*where $\delta$ denotes the hyperbolic distance from $u$ to the geodesic determined by $v$. If $u, v \in H$, then*

$$< u, v >^2 = \begin{cases} \cosh^2\delta, & \text{if the corresponding geodesics are disjoint;} \\ \cos^2\theta, & \text{if the corresponding geodesics intersect,} \end{cases}$$

*where in the first case, $\delta$ is the infimum of hyperbolic distances between points on the two geodesics, and in the second case, $\theta$ is their angle of intersection.*

**Lemma 2.4.** *Given three distinct rays $r_1, r_2, r_3 \subset L^+$ from the origin, there are unique $u_j \in r_j$ so that $< u_j, u_j > = -1$ for $i \neq j$, $i, j = 1, 2, 3$.*

*Proof.* Choose any $v_j \in r_j$ for $i = 1, 2, 3$. We seek $\alpha_j \in \mathbb{R}_{>0}$ so that $< \alpha_j v_j, \alpha_j v_j > = -1$ for $i \neq j$, i.e., $\alpha_j \alpha_j = - < v_j, v_j >^{-1}$, and there is a unique positive solution, namely,

$$\alpha_j = \sqrt{\frac{- < v_j, v_k >}{< v_j, v_j >< v_j, v_k >}}, \text{ for } \{i, j, k\} = \{1, 2, 3\}.$$

□

**Corollary 2.5.** *The group $SO^+(2, 1)$ acts simply transitively on positively oriented triples of distinct rays in $L^+$.*

This is the version in Minkowski space of the familiar 3-effectiveness of the action of the Möbius group on triples of positively oriented points in the circle at infinity. In particular, it follows that any Möbius transformation fixing three points at infinity is necessarily the identity. We remark that we shall use Corollary 2.5 without apology to normalize subsequent calculations and directly compute various quantities with a decidedly nineteenth-century ethos.

We finally come to a definition that is basic for our considerations. A "horocycle" in $\mathcal{U}$ is either a Euclidean circle tangent to the real axis or a horizontal Euclidean line parallel to the real axis. Applying the Cayley transform, a horocycle in $\mathbb{D}$ is thus a Euclidean circle tangent to $S^1_\infty$.

This definition leaves something to be desired since we relied on the underlying Euclidean metric of the Riemannian manifold $\mathcal{U}$, so here is a better and more invariant definition: Choose a point $p$ in the hyperbolic plane and a tangent direction $v$ at $p$, and consider a family of hyperbolic circles whose radius and center diverge in such a controlled manner as to pass through $p$ with tangent direction $v$; such a sequence of hyperbolic circles has a well-defined limit, defined to be a *horocycle*. (Of course, an analogous family of Euclidean circles in the Euclidean plane limits to a Euclidean line, so the existence of horocycles is truly a new phenomenon in hyperbolic geometry.) Thus, a horocycle in $\mathbb{D}$ is indeed a Euclidean circle tangent to $S^1_\infty$, and the point of tangency is called the *center* of the horocycle, with a similar remark and definition for $\mathcal{U}$.

One can check that yet another invariant definition is that a horocycle is a smooth curve in the hyperbolic plane of constant geodesic curvature one.

A direct calculaton using the formulas given earlier allow the calculation of horocycles in Minkowski space, and one finds:

**Lemma 2.6.**

$$\begin{aligned} L^+ &\rightarrow \{\text{horocycles in } \mathbb{H}\} \\ u &\mapsto h(u) = \{v \in \mathbb{H} :< u, v >= -1/\sqrt{2}\} \end{aligned}$$

*establishes an isomorphism between points of* $L^+$ *and the collection of all horocycles in* $\mathbb{H}$. *Furthermore, the center of the corresponding horocycle* $\bar{h}(u)$ *in* $\mathbb{D}$ *is* $\bar{u} \in S^1_\infty$, *and the Euclidean radius of* $\bar{h}(u)$ *in* $\mathbb{D}$ *is* $\frac{1}{1+z\sqrt{2}}$, *where* $u = (x, y, z)$.

The funny choice of constant $-1/\sqrt{2}$ in Lemma 2.6 will be explained later; any negative constant would do just as well here (and this normalization differs from that in our papers).

Let us finally re-consider the action of $A \in SO^+(2,1) \approx PSL_2(\mathbb{R})$ on Minkowski space now armed with this notion of horocycle. As a unimodular linear map acting on $\mathbb{R}^3$, $A$ has at most three non-zero eigenvalues, and there are various cases:

- $A$ is hyperbolic if it has an eigenvalue $\lambda$ with $|\lambda| \neq 1$, so $\lambda$ is real and positive with corresponding simple eigenvector (ray) contained in $L^+$; there is one other eigenvector contained in $L^+$ with eigenvalue $\lambda^{-1}$, and these two eigenvectors correspond to the ideal points at infinity of the invariant geodesic; there is a third eigenvector on $H$ with eigenvalue 1 which corresponds to the invariant geodesic;

• $A$ is parabolic if there is a unique eigenvector contained in $L^+$ with eigenvalue 1 and no eigenvector on $\mathbb{H}$; by Lemma 2.6, there is a corresponding foliation of $\mathbb{H}$ by horocycles which is leafwise invariant under $A$;

• $A$ is elliptic if all its eigenvalues lie on the unit circle with a unique eigenvector in $\mathbb{H}$ corresponding to the unique fixed point of $A$ in $\mathbb{H}$.

## 3. The Farey tesselation and Gauss product

Before pressing forward with geometry and Teichmüller theory, we cannot resist digressing here to briefly discuss topics from algebraic number theory. The first part of this lecture is explicated in [18], and the appendix [59] treats the latter part of this lecture in much greater detail with complete proofs and with a discussion of the background number theory from first principles. See [11] for a detailed treatment of quadratic forms.

Let us begin in the upper half-space model $\mathcal{U}$ and consider the horocycle $h_n$ of Euclidean diameter one centered at $n \in \mathbb{Z} \subset \mathbb{R}$, for each $n \in \mathbb{Z}$. Thus, $h_n$ is tangent to $h_{n\pm1}$ and is disjoint from the remaining horocycles. We also add the horocycle $h_\infty$ centered at infinity given by the horizontal line at height one, which is tangent to each $h_n$.

Two consecutive horocycles $h_n, h_{n+1}$ determine a triangular region bounded by the interval $[n, n+1] \subset \mathbb{R}$ together with the horocyclic segments connecting the centers of the horocycles to the point of tangency of $h_n$ and $h_{n+1}$. There is a unique horocycle contained in such a triangular region which is tangent to $h_n$ and $h_{n+1}$ as well as tangent to the the real axis, and we let $h_{n+\frac{1}{2}}$ denote this horocycle, which is evidently tangent to the real axis at the half-integer point $n + \frac{1}{2}$ and of Euclidean diameter $\frac{1}{4}$. We may continue recursively in this manner, adding new horocyles tangent to the real axis and tangent to pairs of consecutive tangent horocycles in order to produce a family of horocycles $\mathcal{H}$ in $\mathcal{U}$. See Figure 5a.

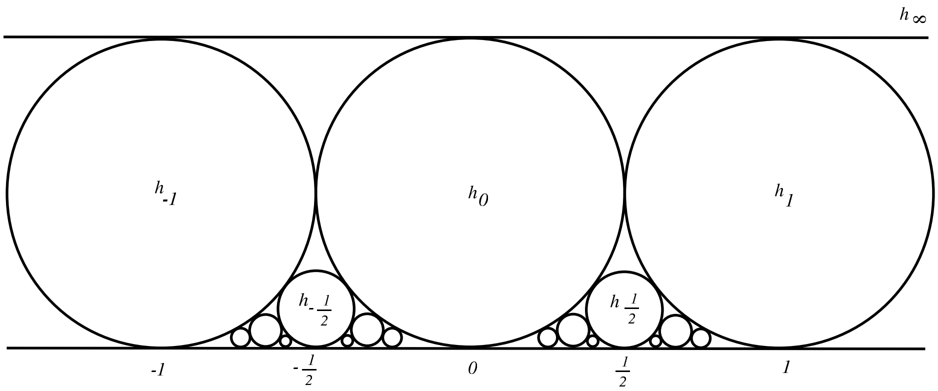



**Figure 5a** Horocyclic packing of upper half-space.

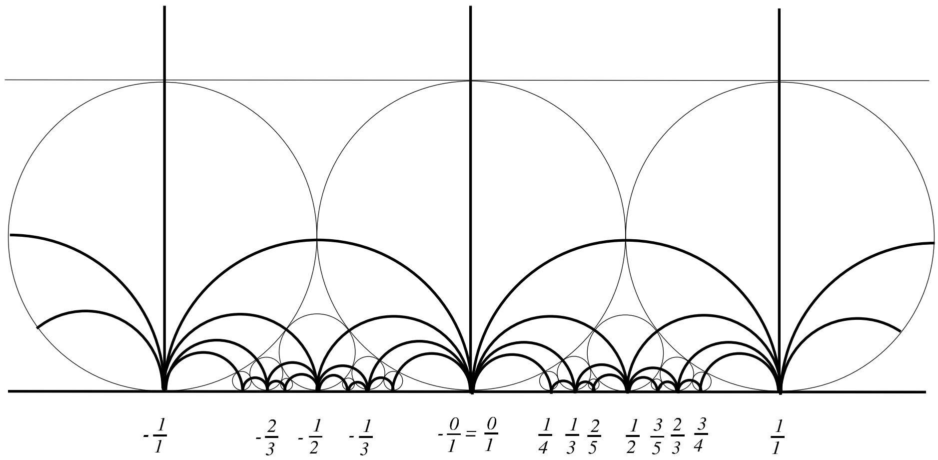



**Figure 5b** The Farey tesselation in upper half-space.

**Lemma 3.1.** *There is a unique horocycle in $\mathcal{H}$ centered at each rational point $\mathbb{Q} \subset \mathbb{R}$, and the horocycle centered at $\frac{p}{q} \in \mathbb{Q}$ has Euclidean diameter $\frac{1}{q^2}$, where $\frac{p}{q}$ is written in reduced form except with $n \in \mathbb{Z}$ written as $\frac{n}{1}$. Furthermore, the horocycles in $\mathcal{H}$ centered at distinct points $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ are tangent to one another if and only if $ps - qr = \pm 1$, and in this case, the horocycle in $\mathcal{H}$ tangent to these two horocycles is centered at $\frac{p+r}{q+s} \in \mathbb{Q}$.*

It is not hard to prove this lemma inductively starting with the second sentence. In fact, this result was discovered by the mineralogist John Farey and solved the long-standing problem of giving a one-to-one enumeration of the rational numbers. After Farey published his empirical findings, Cauchy essentially immediately supplied the proofs.

Now define the *Farey tesselation* to be the collection of hyperbolic geodesics in $\mathcal{U}$ that connect centers of tangent horocyles in $\mathcal{H}$; see Figure 5b. Thus, the Farey tesselation of $\mathcal{U}$ is a countable collection of geodesics that decompose $\mathcal{U}$ into regions called *ideal triangles*, i.e., regions bounded by three geodesics pairwise sharing ideal points at infinity. Figure 6 illustrates the Farey tesselation in $\mathbb{D}$, i.e., the image under the Cayley transform, which we shall denote $\tau_*$ and regard as a set of geodesics in $\mathbb{D}$.

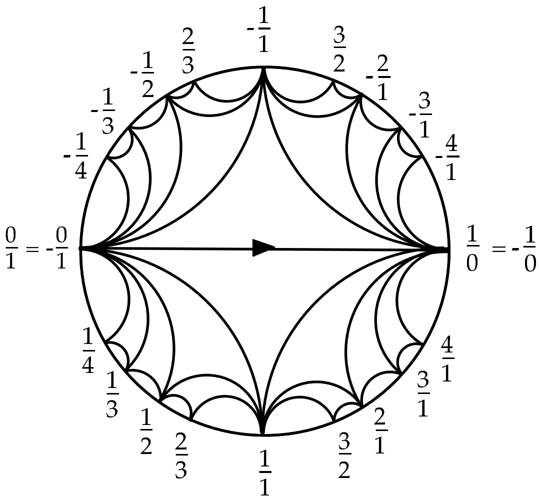



**Figure 6** The Farey tesselation in the Poincaré disk.

The Möbius group $PSL_2(\mathbb{R})$ contains the discrete group $PSL_2(\mathbb{Z})$ consisting of all two-by-two *integral* matrices of determinant one again modulo the equivalence relation identifying the matrix $A$ with its negative $-A$. This subgroup is called the *(classical) modular group* and plays a basic role in number theory, as we shall partly explain in this lecture.

**Lemma 3.2.** *The modular group leaves invariant the Farey tesselation $\tau_*$, mapping geodesics in $\tau_*$ to geodesics in $\tau_*$ and complementary ideal triangles to complementary ideal triangles, and any Möbius transformation leaving invariant $\tau_*$ in this manner lies in the modular group. Furthermore, the modular group acts simply transitively on the oriented edges of $\tau_*$. A generating set is given by any pair of*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ U = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

*where $T^{-1} = SUS$ and $U^{-1} = STS$; a presentation in the generators $S, T$ is given by $S^2 = 1 = (ST)^3$, so the modular group is abstractly the* *free product* $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$. *A fundamental domain for the action of the modular group on* $\mathcal{U}$ *is given by* $\{x+iy \in \mathcal{U} : x^2+y^2 > 1 \text{ and } |x| < \frac{1}{2}\}$.

An especially beautiful combinatorial fact is that the continued fraction expansion of a rational number $\frac{p}{q}$ can be read off from the sequence of right or left turns in the Farey tesselation connecting the point $i$ to $\frac{p}{q}$. Let us just illustrate with an example, the rational $\frac{5}{13}$. Starting from $i$ one makes the following sequence of turns right (R) or left (L) in triangles of $\tau_*$ to arrive at $\frac{5}{13}$: RRLRL. Reading off the number of consecutive turns 2(R), 1(L), 1(R), 1(L), we calculate the continued fraction expansion:

$$\frac{5}{13} = \cfrac{1}{\mathbf{2} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{1} + 1}}}}.$$

The proof of this general fact (which follows from Lemmas 3.1-3.2 and the definition of continued fractions) is left to the reader.

One number-theoretic aspect of the current discussion involves so-called *elliptic curves*, namely, discrete subgroups $\Lambda$ of $\mathbb{C}$ of rank two, also called *lattices*. The quotient of $\mathbb{C}$ by $\Lambda$ is a flat torus together with a distinguished point corresponding to $0 \in \mathbb{C}$, and up to conformal equivalence of this torus, we may take $\Lambda$ to be generated by the unit $1 \in \mathbb{C}$ and a complex number $\tau$ of positive imaginary part, i.e., the analogue of Teichmüller space for elliptic curves is precisely the upper half-plane $\mathcal{U}$. The mapping class group of this torus-with-distinguished-point is precisely the modular group $PSL_2(\mathbb{Z})$ acting by fractional linear transformation on $\mathcal{U}$, and the analogue of Riemann's moduli space is thus $\mathcal{U}/PSL_2(\mathbb{Z})$, which is called the *modular curve* and is illustrated in Figure 7. It is not quite a manifold, rather, it is an *orbifold*, namely, neighborhoods of its point are modeled on quotients of Euclidean space modulo a finite group action, in this case, the finite groups being the trivial group, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z}$.

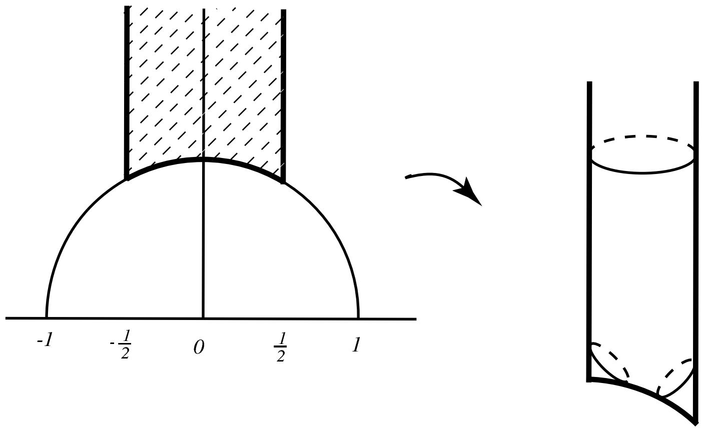



**Figure 7** The modular curve.

In fact, we may remove the distinguished point of the elliptic curve to produce a once-punctured torus, whose Teichmüller space is $\mathcal{U}$, whose mapping class group is $PSL_2(\mathbb{Z})$, and whose moduli space is the modular curve as we shall see in Lecture 5. The modular curve is the unique moduli space of a surface which is also a surface or at least a two-dimensional orbifold.

We shall only require the Farey tesselation $\tau_*$ and its invariance under the modular group $PSL_2(\mathbb{Z})$ in the sequel and have discussed these other aspects just because they are interesting and beautiful.

Our principal topic for this lecture is the "Gauss product" of suitable binary quadratic forms, to which we finally turn our attention, and since this material is developed from first principles in the appendix, our discussion here will just serve to introduce the main ideas (as well as include a basic remark we neglected to mention explicitly in the appendix).

Recall the space $\mathcal{B}_2(\mathbb{R})$ of symmetric real bilinear pairings discussed in the previous lecture, where corresponding to the point $(x, y, z)$ in Minkowski 3-space, we have considered the pairing

$$\begin{pmatrix} z-x & y \\ y & z+x \end{pmatrix} \in \mathcal{B}_2(\mathbb{R})$$

with its corresponding quadratic form $(z-x)\xi^2 + 2y\xi\eta + (z+x)\eta^2$, where $\xi, \eta$ denote indeterminates.

We shall now restrict our attention to the subspace $\mathcal{B}_2(\mathbb{Z})$ corresponding to *integral* binary forms and shall let the integral quadratic form $a\xi^2 + b\xi\eta + c\eta^2$, where $a, b, c \in \mathbb{Z}$, be denoted simply $[a, b, c]$. We say that $[a, b, c]$ is *primitive* if $a, b, c$ have no common divisors, i.e., $gcd\{a, b, c\} = 1$, and define the *discriminant* of $[a, b, c]$ to be

$D = b^2 - 4ac$. In particular, $b$ is even if and only if $D$ is equivalent to zero modulo four, and $b$ is odd if and only if $D$ is equivalent to one modulo four. The quadratic form $[a, b, c]$ is said to be *definite* or *imaginary* if $D < 0$, and it is said to be *indefinite* or *real* if $D > 0$; otherwise if $D = 0$, it is said to be *degenerate*. The modular group acts on integral quadratic forms by change of basis, as in the previous lecture, and leaves invariant both primitivity of the form as well as the discriminant as one easily confirms. We let $[\![a, b, c]\!]$ denote the corresponding orbit and consider the set

$$\mathcal{G}(D) = \{\, [\![a, b, c]\!] \; : [a, b, c] \text{ is primitive of discriminant } D\}.$$

Gauss showed that for each discriminant $D$, $\mathcal{G}(D)$ has the structure of a finite abelian group, and we shall formulate a version of this group law presently. (It is an important open problem to explicitly calculate the orders of these groups, the so-called "class numbers", cf. [11].)

We say that two classes of integral quadratic forms $[f_1], [f_2]$ are *unitable* if their respective discriminants $D_1, D_2$ satisfy $D_j = t_j^2 d$ for some $t_j \in \mathbb{Z}$, i.e., if $D_1 D_2$ is the square of some integer, and in this case, we may define $t'_j = t_j / gcd\{t_1, t_2\}$. Two unitable forms $f_1, f_2$ are said to be *concordant* if there are respective representatives of their $PSL_2(\mathbb{Z})$-orbits of the form

$$[a_1, t'_1 b, {t'_1}^2 a_2 c] \text{ and } [a_2, t'_2, {t'_2}^2 a_1 c].$$

Given concordant forms $f_1, f_2$, we define the *(Gauss) product* of their respective classes $[f_1], [f_2]$ to be

$$[f_1][f_2] = [\![a_1 a_2, b, c]\!] \,.$$

**Theorem 3.3.** *Given two unitable classes, there exist concordant representatives, and the class of the product is well-defined. Fixing a square-free discriminant d and setting $\mathcal{S}(d) = \coprod_{t \geq 1} \mathcal{G}(t^2 d)$, the product gives $\mathcal{S}(d)$ the structure of an abelian semigroup, and the restriction of the product gives $\mathcal{G}(t^2 d) < \mathcal{S}(d)$ the structure of a finite abelian group, for each $t \geq 1$.*

As we shall see, the elementary yet crucial geometric point about concordant forms is:

**Lemma 3.4.** *Suppose that $[a_j, b_j, c_j]$, for $i = 1, 2$, are primitive forms of respective discriminants $D_1, D_2$, where $b_1 b_2 \geq 0$. Then the two forms are concordant if and only if the following two conditions hold:*

- $D_1 b_2^2 = D_2 b_1^2$, *and thus* $D_j = t_j^2 d$ *for some* $t_j \in \mathbb{Z}$, *and we define* $b_j = bt'_j$, *for* $i = 1, 2$;
- $4a_1a_2 | b^2 - d(gcd\{t_1, t_2\})^2$.

Turning attention now to the case of definite quadratic forms (which we should remark is the "easy" case of quadratic forms, where our understanding is much more complete), the key geometric point is that integral points $(x, y, z)$ of Miknowski 3-space which lie inside of $L^+$ (i.e., have positive $z$-coordinate and negative Minkowski length) at once correspond to definite integral quadratic forms $[z - x, 2y, z + x]$ as above and to suitable points of $\mathbb{H}$, namely, the ray from the origin in Minkowski 3-space through the integral point $(x, y, z)$ inside $L^+$ meets $\mathbb{H}$ in the point $1/(z^2 - x^2 - y^2)(x, y, z)$. The $PSL_2(\mathbb{Z})$-orbit $[\![ z - x, 2y, z + x ]\!]$ thus corresponds to a point of the modular curve, so the Gauss product defines an abelian group structure on appropriate subsets of the modular curve. We ask (and answer) whether this Gauss product can be understood geometrically in these terms.

The critical property of concordant forms is that the first condition in Lemma 3.4 is projectively invariant, i.e., it is defined on projective classes of definite quadratic forms and hence determines some condition on points of $\mathbb{H}$.

By definition, a definite quadratic form $[a, b, c]$ has exactly two roots of $az^2 + bz + c = 0$, and exactly one of these roots

$$\omega_{[a,b,c]} = -\frac{b}{2a} + i\sqrt{\frac{-D}{4a^2}} = \frac{p}{q} + i\sqrt{\frac{r}{s}} \in \{z = x + iy : x, y^2 \in \mathbb{Q} \text{ and } y > 0\},$$

called the *primitive root*, lies in $\mathcal{U}$. (One can identify elliptic curves with quadratic forms by identifying the primitive root with the invariant $\tau$ of the elliptic curve discussed above, and then this locus of primitive roots of definite integral quadratic forms corresponds to the collection of elliptic curves that "admit complex multiplication", cf. [12].)

**Lemma 3.5.** *Given an integral point* $(x, y, z)$ *of Minkowski 3-space inside* $L^+$ *with corresponding point* $v \in \mathbb{H}$, *the Cayley transform in* $\mathcal{U}$ *of the central projection* $\bar{v} \in \mathbb{D}$ *agrees with the primitive root* $\omega_{[z-x,2y,z+x]}$. *Furthermore, given a point* $\omega = \frac{p}{q} + i\sqrt{\frac{r}{s}}$, *the primitive form proportional to* $[q^2 s, -2pqs, p^2 s + q^2 r]$ *has* $\omega$ *as its primitive root.*

The proof is a somewhat involved but elementary calculation left to the reader (using the formulas in the previous section); it is the first part of this lemma that we neglected to explicitly mention in the appendix. We are led to consider the level sets of $D/a^2, D/c^2$ and especially $D/b^2$, for if two primitive definite forms lie on a common level set of one of these functions, then they are unitable. Setting $\omega = \frac{p}{q} + i\sqrt{\frac{r}{s}} = u + iv \in \mathcal{U}$, we compute from Lemma 3.5 that

- $D/a^2 = -\alpha^2 \;\Leftrightarrow\; \frac{r}{s} = \frac{\alpha^2}{4} \;\Leftrightarrow\; v = \frac{\alpha}{2}$;
- $D/b^2 = -\beta^2 \;\Leftrightarrow\; \frac{r}{s} = \beta^2\,\frac{p^2}{q^2} \;\Leftrightarrow\; v = \pm\beta u$;
- $D/c^2 = -\gamma^2 \;\Leftrightarrow\; \frac{p^2}{q^2}\sqrt{\frac{s}{r}} + \sqrt{\frac{r}{s}} = 2\gamma^{-1} \;\Leftrightarrow\; u^2 + (v - \gamma^{-1})^2 = \gamma^{-2}$,

where $\alpha, \beta, \gamma > 0$. These respective loci in $\mathcal{U}$ are thus horizontal lines, rays from the origin, and circles tangent to $\mathbb{R}$ at zero as illustrated in Figure 8. The first and last cases are the familiar horocycles centered at zero and infinity, and the second case is new to us and bears further discussion.

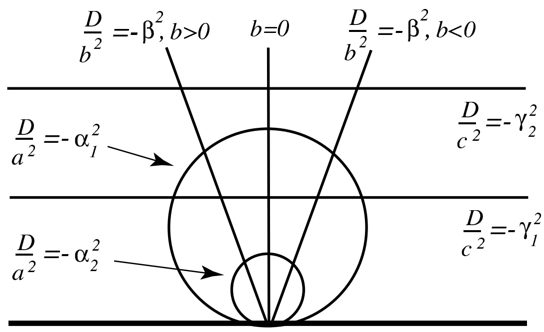



**Figure 8** Horocycles and hypercycles in upper half-space.

An $\delta$-*hypercycle* to a geodesic in $\mathcal{U}$ is a component of the locus of points at distance $\delta \geq 0$ from the geodesic, and there are thus two $\delta$-hypercycles for each $\delta > 0$ while the 0-hypercycle is just the geodesic itself. Put another way, one can check that a hypercycle is a locus of constant geodesic curvature between zero and one, and one can thus think of hypercycles as interpolating between geodesics (of curvature zero) and horocycles (of curvature one). In particular, hypercycles to the imaginary ray in $\mathcal{U}$ are precisely Euclidean rays from the origin, i.e., the second case above corresponds to hypercycles. Projections to the modular curve of horocycles centered at infinity or of hypercycles to the imaginary ray will be called simply *horocycles* and *hypercycles* in the modular curve as illustrated respectively on the right and left in Figure 9.

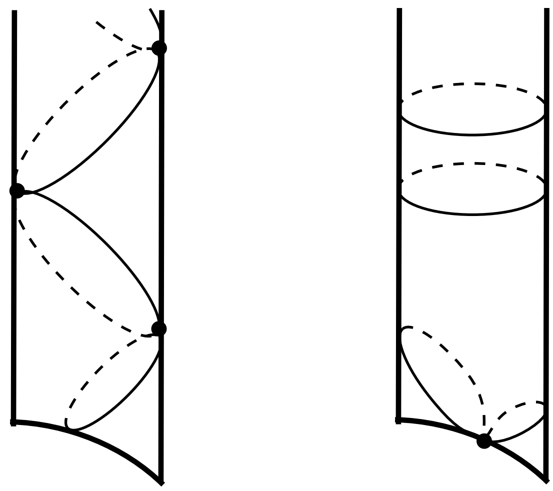


**Figure 9** One hypercycle and several horocycles in the modular curve.

**Corollary 3.6.** *Two classes of definite primitive quadratic forms are unitable if and only if they lie on a common hypercycle in the modular curve. Furthermore, if the classes lie on a common horocycle in the modular curve, then they are unitable as well.*

It is worth emphasizing that two primitive definite quadratic forms on a common hypercycle may not be concordant for *that* hypercycle, i.e., the second condition in Lemma 3.4 does not follow from the first.

We say that a definite form $f$ *translates* to another form $f'$ if there is some element $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ of the modular group sending $f$ to $f'$. Here is our promised geometric description of the Gauss product for definite forms:

**Theorem 3.7.** *Suppose that $f_1, f_2$ are concordant primitive definite forms with corresponding hypercycle $h$. Then the product $[f_1][f_2]$ is represented by the point $f \in h$ closest to the origin with the property that whenever $f_1$ and $f_2$ translate, possibly by different elements of the modular group, to concordant forms on a common hypercycle $h'$, then $f$ also translates to $h'$.*

It would be most interesting to formulate a corresponding geometric description of the Gauss product for indefinite forms. In this case, the natural geometric invariant of an indefinite form $[a, b, c]$ is the hyperbolic geodesic connecting the two real roots of $az^2 + bz + c = 0$.

## 4. Basic formulas

Throughout this lecture, which is principally based on [53] but also includes calculations from [42, 58], we shall conveniently normalize our calculations using Corollary 2.5, and to this end, we introduce a convenient basis for Minkowski 3-space as follows. Define the *standard light-cone basis*

$$
\begin{aligned}
u &= \frac{1}{\sqrt{2}}\,(-1, 0, 1),\\
v &= \frac{1}{\sqrt{2}}\,(1, 0, 1),\\
w &= \sqrt{2}\,(0, -1, 1),
\end{aligned}
$$

so that $u, v, w \in L^+$, $< u, v > = < u, w > = < v, w > = -1$, and

$$u \mapsto 0,\ v \mapsto \infty,\ w \mapsto 1$$

under the Cayley transform. Furthermore, the standard basis vectors in $\mathbb{R}^3$ are expressed in the light-cone basis as

$$(1, 0, 0) = \frac{1}{\sqrt{2}}(v - u),\ (0, 1, 0) = \frac{1}{\sqrt{2}}(u + v - w),\ (0, 0, 1) = \frac{1}{\sqrt{2}}(v + u).$$

Given a pair of horocycles $h_1, h_2$, say in $\mathbb{D}$, with distinct centers, consider the geodesic $\gamma$ in $\mathbb{D}$ connecting their centers. Let $\delta$ denote the signed hyperbolic distance along $\gamma$ between the points $h_1 \cap \gamma$ and $h_2 \cap \gamma$, where the sign of $\delta$ is taken to be positive if and only if $h_1$ and $h_2$ are disjoint. Define the *lambda length* of $h_1, h_2$ to be

$$\lambda(h_1, h_2) = \sqrt{\exp\,\delta}.$$

These are our basic invariants, and essentially all of our calculations will be performed using them.

Put another way, a *decoration* on a hyperbolic geodesic is the specification of a pair of horocycles, one centered at each ideal point of the geodesic, and the lambda length is an invariant of a decorated geodesic. Thus though a hyperbolic geodesic has infinite length, a decoration gives a way to truncate the geodesic and thereby define a sensible length, the lambda length. These invariants are so fundamental to our treatment that we shall often simply identify a geodesic with its lambda length when no confusion may arise.

**Lemma 4.1.** *Suppose that $u_1, u_2 \in L^+$ do not lie on a common ray in $L^+$ and let $h(u_1), h(u_2)$ be the horocycles corresponding to these points via affine duality in Lemma 2.6. Then the lambda length is given by*

$$\lambda(h(u_1), h(u_2)) \;=\; \sqrt{- < u_1, u_2 >}.$$

The identification of horocycles with points of $L^+$ via affine duality in Lemma 2.6 is therefore also geometrically natural in this sense.

*Proof.* We may normalize using the standard light-cone basis so that $u_1 = t_1 u$ and $u_2 = t_2 v$ for some $t_1, t_2 \in \mathbb{R}_{>0}$. We seek the points $\zeta_j \in h(u_j)$, for $j = 1, 2$, on the geodesic connecting the centers of $h(u_1), h(u_2)$, i.e., we seek $\zeta_j = x_j u + y_j v$ so that

$$-t_1 y_1 \;= < \zeta_1, t_1 u > = \; -\frac{1}{\sqrt{2}} \; = < \zeta_2, t_2 v > = \; -t_2 x_2$$

and $\zeta_1, \zeta_2 \in \mathbb{H}$, i.e.,

$$-2x_j y_j \;= < \zeta_j, \zeta_j > = \; -1. \text{ for } j = 1, 2.$$

Solving these equations, we find

$$\zeta_1 = \frac{t_1}{\sqrt{2}}\, u \;+\; \frac{1}{\sqrt{2}t_1}\, v,$$
$$\zeta_2 = \frac{1}{\sqrt{2}t_2}\, u \;+\; \frac{t_2}{\sqrt{2}}\, v.$$

Thus from Lemma 2.3, we have

$$\cosh^2\delta \;= < \zeta_1, \zeta_2 >^2 = \; \big[\frac{1}{2}\,(t_1 t_2 + (t_1 t_2)^{-1})\big]^2,$$

so $< u_1, u_2 > = < t_1 u, t_2 v > = \; -t_1 t_2$ gives

$$\exp\; \pm\delta \;=\; - < u_1, u_2 > .$$

Since $\delta \to \infty$ as $t_1 \to \infty$ or $t_2 \to \infty$ by the last part of Lemma 2.6, we must take the plus sign, completing the proof. $\square$

Armed with this result, direct calculation in the upper half-plane yields:

**Corollary 4.2.** *Given horocycles $h_j$ in $\mathcal{U}$ with distinct centers $x_j \in \mathbb{R}$ of respective Euclidean diameters $\Delta_j$, for $j = 1, 2$, the lambda length is given by*

$$\lambda(h_1, h_2) \;=\; \frac{|x_1 - x_2|}{\sqrt{\Delta_1 \Delta_2}};$$

*likewise, given the hororcycle $h$ centered at infinity of height $H$, the lambda length is given by*

$$\lambda(h, h_1) = \sqrt{\frac{H}{\Delta_1}}.$$

*Furthermore, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Möbius transformation with $Ah_1$ centered at $Ax_1 \in \mathbb{R}$, then $Ah_1$ has Euclidean diameter $\Delta_1/(cx_1 + d)^2$; likewise, if $Ax_1 = \infty$, then $Ah_1$ has height $\Delta_1^{-1}/c^2$.*

We conclude that Euclidean diameters of horocycles in upper half-space scale by derivatives of Möbius transformations (which will serve to motivate a later definition, cf. Theorem 8.8).

The lambda length is thus an invariant of a pair of horocycles or of a decorated geodesic, and we turn our attention next to triples of horocycles. Just as in Lemma 2.4, given a triple of positive numbers $a, b, e$, there are unique positive multiples $\alpha u, \beta v, \gamma w$ of the standard light-cone basis realizing

$$< \alpha u, \beta v > = \; -e^2, \; < \alpha u, \gamma w > = \; -a^2, \; < \beta v, \gamma w > = -b^2,$$

namely,

$$\alpha = \frac{ae}{b}, \; \beta = \frac{be}{a}, \; \gamma = \frac{ab}{e}.$$

**Corollary 4.3.** *Triples of lambda lengths give a parametrization of Möbius orbits of triples of horocycles with distinct centers.*

**Lemma 4.4.** *Given a triple of horocycles $h_1, h_2, h_3$ with distinct centers, let $\lambda_j = \lambda(h_k, h_\ell)$ denote the lambda lengths and let $\gamma_j$ denote the geodesic connecting the centers of $h_k, h_\ell$, for $\{j, k, \ell\} = \{1, 2, 3\}$. Then the hyperbolic length of the horocyclic segment in $h_j$ between $\gamma_k$ and $\gamma_\ell$ is given by $\frac{\lambda_j}{\lambda_k \lambda_\ell}$.*

*Proof.* Again, we use Corollary 2.5 and the earlier calculation to arrange that the three points in $L^+$ corresponding to the triple of horocycles are $\frac{ae}{b}u$, $\frac{be}{a}v$, $\frac{ab}{e}w$. We seek the point $\zeta = xu + yv \in \mathbb{H}$ so that

$$-\frac{1}{\sqrt{2}} \;= <\zeta, \frac{be}{a}v> = \;\frac{be}{a}\; <\zeta, v> = \;-\frac{be}{a}\;x,$$

i.e. $x = \frac{1}{\sqrt{2}}\,\frac{a}{be}$. Now using $-1 \;= <\zeta,\zeta>$, we find $y = \frac{1}{2x}$, so

$$\zeta \;=\; \frac{a}{be}\;u + \frac{be}{2a}\;v \;=\; (-\frac{a}{2be} + \frac{be}{2a}, 0, \frac{a}{2be} + \frac{be}{2a})$$

as a vector in $\mathbb{R}^3$. Use the explicit mapping $\mathbb{H} \to \mathcal{U}$ given by $(x, y, z) \mapsto (\frac{y}{z-x}, \frac{1}{z-x})$ to compute the imaginary part of the image of $\zeta$ to be $\frac{be}{a}$; since $u, v, w$ respectively map to $0, \infty, 1$, the hyperbolic segment lying in $h(\frac{be}{a}v)$ maps to the horizontal segment of Euclidean length one at height $\frac{be}{a}$, which has hyperbolic length $\frac{a}{be}$ using the expression for the hyperbolic metric in $\mathcal{U}$. The other formulas follow by symmetry. $\square$

A *decoration* on an ideal triangle is a triple of horocycles, one centered at each ideal point of the triangle. Define a *sector* to be an end of an ideal triangle, so associated with a decorated ideal triangle, each sector has a corresponding horocyclic arc as above, whose length is called the *h-length* of the sector. We have just shown that the h-length of a sector is the opposite lambda length divided by the product of adjacent lambda lengths, and it was to guarantee this formula that we took the funny constant $-1/\sqrt{2}$ in Lemma 2.6. (We should also mention that this corrects Proposition 2.8 in [53].)

It follows from Lemma 4.4 that the product of the h-lengths of two sectors in a decorated ideal triangle is the reciprocal of the square of the lambda length of the decorated geodesic connecting the sectors, i.e., in the notation of Lemma 4.4, we have $\frac{\lambda_j}{\lambda_k\lambda_\ell}\,\frac{\lambda_k}{\lambda_j\lambda_\ell} \;=\; \frac{1}{\lambda_\ell^2}$.

**Lemma 4.5.** *Given $u_1, u_2, u_3 \in L^+$ no two of which lie on a common ray in $L^+$, define $\lambda_j = \lambda(h(u_k), h(u_\ell))$, for $\{j, k, \ell\} = \{1, 2, 3\}$. The affine plane containing $u_1, u_2, u_3$ is elliptic if and only if the strict triangle inequalities hold among $\lambda_1, \lambda_2, \lambda_3$, it is parabolic if and only if some triangle equality holds, and it is hyperbolic if and only if some weak triangle inequality fails.*

*Proof.* The tangent space to the affine plane is spanned by $v_1 = u_1 - u_3$ and $v_2 = u_2 - u_3$, and we compute that $< v_j, v_j > = \; 2\lambda_k^2$, for $\{j, k\} =$

$\{1,2\}$ while $< v_1, v_2 >= \lambda_1^2+\lambda_2^2-\lambda_3^2$. The determinant of the Minkowski pairing restricted to the affine plane is thus

$$\begin{aligned} < v_1, v_1 > < v_2, v_2 > \ - \ < v_1, v_2 >^2 & \\ &= 4\lambda_1^2\lambda_2^2 - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2)^2 \\ &= -\lambda_1^4 - \lambda_2^4 - \lambda_3^4 + 2\lambda_1^2\lambda_2^2 + 2\lambda_1^2\lambda_3^2 + 2\lambda_2^2\lambda_3^2 \\ &= (\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2 - \lambda_3)(\lambda_1 + \lambda_3 - \lambda_2)(\lambda_2 + \lambda_3 - \lambda_1), \end{aligned}$$

and the result then follows from Lemma 2.1 since at most one factor in the last expression can be non-positive for $\lambda_1, \lambda_2, \lambda_3 > 0$. □

**Lemma 4.6.** *Given $u_1, u_2, u_3 \in L^+$, define $\lambda_j = \lambda(h(u_k), h(u_\ell))$, for $\{j, k, \ell\} = \{1, 2, 3\}$ and let $h_1, h_2, h_3$ denote the corresponding horocycles which we assume have pairwise distinct centers. Then there is a point equidistant to $h_1, h_2, h_3$ if and only if $\lambda_1, \lambda_2, \lambda_3$ satisfy the strict triangle inequalities, and in this case, the equidistant point $\zeta$ is unique. Furthermore in this case, let $\alpha_j$ denote the h-length of the sector corresponding to $h_j$ and let $\gamma_j$ be the geodesic connecting the centers of $h_k, h_\ell$, for $\{j, k, \ell\} = \{1, 2, 3\}$. Then the geodesic connecting $\zeta$ to the center of $h_j$ is at signed distance $\frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3) - \alpha_\ell$ from $\gamma_k$ along $h_j$, where the sign is positive if and only if $\zeta$ lies on the same side of $\gamma_k$ as the center of $h_\ell$, for $\{j, k, \ell\} = \{1, 2, 3\}$.*

*Proof.* As usual, we shall compute in the standard light-cone basis $u, v, w$ and arrange that the three points in $L^+$ corresponding to the triple of horocycles are $\frac{ae}{b}u, \frac{be}{a}v, \frac{ab}{e}w$. We seek $\zeta = xu + yv + zw \in \mathbb{H}$ so that

$$< \zeta, \frac{ae}{b}u > = < \zeta, \frac{be}{a}v > = < \zeta, \frac{ab}{e}w > .$$

Define $A = y + z, B = x + z, C = x + y$, so we must solve the equation

$$2 = A(B + C - A) + B(A + C - B) + C(A + B - C)$$

where $A = \frac{b^2}{e^2}C$ and $B = \frac{a^2}{e^2}C$. Thus,

$$\begin{aligned} 2 &= C^2\big[\frac{b^2}{e^2}(\frac{a^2}{e^2} + 1 - \frac{b^2}{e^2}) + \frac{a^2}{e^2}(\frac{b^2}{e^2} + 1 - \frac{a^2}{e^2}) + (\frac{b^2}{e^2} + \frac{a^2}{e^2} - 1)\big] \\ &= C^2/e^4\ \big[2a^2b^2 + 2a^2e^2 + 2b^2e^2 - a^4 - b^4 - e^4\big] \\ &= C^2/e^4\ (a + b + e)(a + b - e)(a + e - b)(b + e - a), \end{aligned}$$

so there is indeed a solution if and only if $a, b, e$ satisfy all three possible strict triangle inequalities. Furthermore in this case setting

$$K = (a + b + e)(a + b - e)(a + e - b)(b + e - a),$$

we find

$$(A,B,C) = \pm\sqrt{\frac{2}{K}}(b^2,a^2,e^2)$$

and must take the positive sign to guarantee a positive $z$-coordinate in Minkowski 3-space. Thus, the unique equidistant point is given by

$$\zeta = \sqrt{\frac{1}{2K}}\ \big[(a^2+e^2-b^2)u + (b^2+e^2-a^2)v + (a^2+b^2-e^2)w\big].$$

Let us find a Minkowski normal $n$ for the hyperbolic plane through the origin determining the geodesic $\gamma$ passing through $\zeta$ and $u$. To this end, $< n, u > =\ 0$ gives $n = xu + yv - yw$, and $< n, \zeta > = 0$ further gives

$$\begin{aligned} 0 &= -\ \sqrt{2K}\ \ < n, \zeta > \\ &= (b^2+e^2-a^2)(x-y) + (a^2+b^2-e^2)(x+y) \\ &= 2b^2x + 2(a^2-e^2)y. \end{aligned}$$

A normal to the plane containing $\gamma$ is thus given by

$$n = (e^2-a^2)u + b^2v - b^2w.$$

Next, we seek a point $r = xu+yv+zw \in L^+$ so that the ideal points of $\gamma$ in $\mathbb{D}$ are $\bar r, \bar u \in S^1_\infty$, that is,

$$0\ =\ -< n, r > =\ y(e^2-a^2-b^2) + z(e^2+b^2-a^2),$$

so $r \in L^+$ gives

$$0 = xz\ (b^2+e^2-a^2)\ +\ z^2\ (b^2+e^2-a^2)\ +\ xz\ (a^2+b^2-e^2).$$

Thus, $xz = \frac{a^2-b^2-e^2}{2b^2}\ z^2$, and it follows that such an $r \in L^+$ is given by

$$r = (a^2+b^2-e^2)(a^2-b^2-e^2)\ u\ +\ 2b^2(b^2+e^2-a^2)\ v\ +\ 2b^2(a^2+b^2-e^2)\ w.$$

We may finally compute the length along the horocycle $h(\frac{ae}{b}u)$ between $\gamma \cap h(\frac{ae}{b}u)$ and the geodesic asymptotic to the rays of $u, v$ using Lemma 4.4 and find this length to be

$$\begin{aligned} \pm\sqrt{-\frac{< \frac{be}{a}v, r >}{< \frac{ae}{b}u, r >< \frac{ae}{b}u, \frac{be}{a}v >}} \ &=\ \pm\sqrt{\frac{(a^2+b^2-e^2)^2}{4a^2b^2c^2}} \\ &=\ \pm\frac{1}{2}\ (\frac{a}{be} + \frac{b}{ae} - \frac{e}{ab}). \end{aligned}$$

where the plus sign finally follows by comparison with the expression above for $\zeta$ and the definition of the signed distance. The other formulas follow by symmetry. $\square$

We finally turn our attention to quadruples of horocycles with distinct centers, or put another way, *decorated ideal quadrilaterals*, that is, ideal quadrilaterals with a horocycle centered at each ideal point. There is again a basic calculation from which several results are derived as follows:

**Basic Calculation** Consider multiples $u' = \frac{ae}{b}u, v' = \frac{be}{a}v, w' = \frac{ab}{e}w$ of the standard light-cone basis $u, v, w$ as before. Given two further positive real numbers $c, d$, we claim that there is a unique point $\zeta = xu + yv + zw \in L^+$ so that $< \zeta, u' > = -d^2$, $< \zeta, v' > = -c^2$, and $\zeta, w$ lie on opposite sides of plane through the origin containing $u'$ and $v'$. Indeed, we have the equations

$$\begin{aligned} -d^2 &= < \zeta, u' > = \frac{ae}{b} < \zeta, u > = -\frac{ae}{b}(y+z), \\ -c^2 &= < \zeta, v' > = \frac{be}{a} < \zeta, v > = -\frac{be}{a}(x+z), \end{aligned}$$

which give $y = \frac{bd^2}{ae} - z$, $x = \frac{ac^2}{be} - z$. Now using that $\zeta \in L^+$, i.e., $0 = xy + xz + yz$, we find

$$\begin{aligned} 0 &= \frac{abc^2d^2}{abe^2} + z^2 - z(\frac{ac^2}{be} + \frac{bd^2}{ae}) + (\frac{ac^2}{be} - z)z + (\frac{bd^2}{ae} - z)z \\ &= \frac{c^2d^2}{e^2} - z^2. \end{aligned}$$

Thus, $z = \pm\frac{cd}{e}$, and we must take the minus sign to have $\zeta$ on the correct side of the plane through the origin containing $u'$ and $v'$. It follows that the unique solution is given by

$$\zeta = \frac{c}{eb}(ac+bd)u \; + \; \frac{d}{ea}(ac+bd)v \; - \; \frac{cd}{e}w$$

completing the basic calculation and proving the claim, which is formalized in the next lemma.

**Lemma 4.7.** *Given $u_1, u_2 \in L^+$ and real numbers $\lambda_1\lambda_2, \lambda_2$ satisfying $< u_1, u_2 > = -\lambda_3^2$, there is a unique point $u_3$ on either side of the plane through the origin containing $u_1, u_2$ so that $< u_2, u_3 > = -\lambda_1^2$ and $< u_1, u_3 > = -\lambda_2^2$. Furthermore, the ratio $\lambda_1/\lambda_2$ uniquely determines the ray in $L^+$ containing $u_3$.*

**Corollary 4.8.** *Five-tuples of lambda lengths give a parametrization of Möbius orbits of quadruples of horocycles with distinct centers.*

*Proof.* Given an ideal quadrilateral, choose a diagonal decomposing it into a pair of adjacent ideal triangles. Lambda lengths of the frontier edges of the quadrilateral together with the lambda length of the chosen diagonal uniquely determine the Möbius orbit of a decoration on the ideal quadrilateral by the basic calculation. □

Several other important results also follow from the basic calculation:

**Corollary 4.9.** *Suppose $u_1, u_2, u_3, u_4 \in L^+$, where $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4 \in S^1_\infty$ are distinct and occur in this counter-clockwise cyclic order, and let $\lambda_{jk} = \lambda(u_j, u_k) = \sqrt{- < u_j, u_k >}$ for $j, k = 1, 2, 3, 4$, denote the lambda lengths. Then:*

**a) [Ptolemy's equation]** $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$*;*

**b) [Cross ratio]** *the conformal map sending $\bar{u}_1 \mapsto 0$, $\bar{u}_2 \mapsto 1$, and $\bar{u}_3 \mapsto \infty$ also sends $\bar{u}_4 \mapsto -\frac{\lambda_{23}\lambda_{14}}{\lambda_{12}\lambda_{34}}$;*

**c) [Shear coordinate]** *letting $\gamma$ be the geodesic with ideal points $\bar{u}_1, \bar{u}_3$ and dropping perpendiculars from $\bar{u}_2$ and $\bar{u}_4$ to $\gamma$, the signed distance between the points of intersection with $\gamma$ is given by* $\log \frac{\lambda_{23}\lambda_{14}}{\lambda_{12}\lambda_{34}}$*, where the sign is positive if and only if these points lie to the right of one another along $\gamma$.*

*Proof.* Adopting the notation of the basic calculation, we find

$$- < \zeta, \frac{ab}{e} w > = \frac{ab}{e}[\frac{c}{be}(ac+bd) + \frac{d}{ea}(ac+bd)] = \frac{(ac+bd)^2}{e^2},$$

proving part a).

For part b) again in the notation of the basic calculation, write $\zeta$ as a vector in $\mathbb{R}^3$ as

$$\begin{aligned} \zeta &= \frac{c}{eb}(ac+bd)u + \frac{d}{ea}(ac+bd)v - \frac{cd}{e}w \\ &= (\frac{1}{\sqrt{2}}(\frac{d}{ae} - \frac{c}{be})(ac+bd), \sqrt{2}\,\frac{cd}{e}, \frac{1}{\sqrt{2}}(\frac{d}{ae} + \frac{c}{be})(ac+bd) - \sqrt{2}\,\frac{cd}{e}) \end{aligned}$$

and apply the transformation $\mathbb{H} \to \mathcal{U}$ where $(x, y, z) \mapsto (-y+i)/(z-x)$ to find the real part

$$\frac{-\sqrt{2}\frac{cd}{e}}{-\sqrt{2}\frac{cd}{e} + \sqrt{2}\frac{c}{be}(ac+bd)} = \frac{-bd}{-bd+ac+bd} = -\frac{bd}{ac}.$$

For part c), a Minkowski normal to the plane determining the geodesic asymptotic to $u, v$ is evidently $u + v - w$, so by the last part of Lemma 2.3, if another such Minkowski normal $n = xu + yv + zw$ corresponds to a perpendicular geodesic, then $0 = -< n, u+v-w > = 2z$, and furthermore, $n \in H$ gives $1 = < n, n > = -2xy$. Thus, a perpendicular geodesic has corresponding unit normal of the form $n = xu - \frac{1}{2x}v$.

In particular, the unit normal $n_w$ of the perpendicular geodesic asymptotic to $w$ thus has $0 = < n_w, w > = < xu - \frac{1}{2x}v, w > = \frac{1}{2x} - x$, and so $n_w = \frac{1}{\sqrt{2}}u - \frac{1}{\sqrt{2}}v$. Likewise again in the notation of the basic calculation, the unit normal $n_\zeta$ of the perpendicular geodesic asymptotic to $\zeta$ thus has

$$\begin{aligned} 0 &= < n_\zeta, \zeta > \\ &= < xu - \frac{1}{2x}v, \frac{c}{be}(ac + bd)u \ + \ \frac{d}{ae}(ac + bd)v \ - \ \frac{cd}{e}w > \\ &= \ -x\left[\frac{d}{ae}(ac + bd) - \frac{cd}{e}\right] \ + \ \frac{1}{2x}\left[\frac{c}{be}(ac + bd) - \frac{cd}{e}\right] \\ &= \ -x\left[\frac{bd^2}{ae}\right] \ + \ \frac{1}{2x}\left[\frac{ac^2}{be}\right], \end{aligned}$$

so $x = \pm\frac{1}{\sqrt{2}}\frac{ac}{bd}$, and furthermore comparing with the expression in part b) for the cross ratio and by definition of the vector $u$, we see that we must take the positive sign. Thus, we find that $n_\zeta = \frac{1}{\sqrt{2}}\frac{ac}{bd} - \frac{1}{\sqrt{2}}\frac{bd}{ac}$.

Again applying the last part of Lemma 2.3, the desired distance $\delta$ between the perpendiculars is given by

$$\begin{aligned} \cosh^2\delta &= < n_w, n_\zeta >^2 \\ &= < \frac{1}{\sqrt{2}}u - \frac{1}{\sqrt{2}}v, \frac{1}{\sqrt{2}}\frac{ac}{bd}u - \frac{1}{\sqrt{2}}\frac{bd}{ac}v >^2 \\ &= \ \left[\frac{1}{2}\left(\frac{bd}{ac} + \frac{ac}{bd}\right)\right]^2. \end{aligned}$$

Thus, we find $\delta = \pm\log\frac{bd}{ac}$, and the positive sign follows again from the expression for cross ratios in part b) and the definition of $x$. $\square$

Notice the similarity of Corollary 4.9a with Ptolemy's classical theorem that a Euclidean quadrilateral inscribes in a circle if and only if the product of diagonal lengths is the sum of products of opposite lengths. Indeed, a more conceptual proof of Corollary 4.9a is as follows. Notice that the asserted formula is in fact independent under scaling each of the points $u_1, u_2, u_3, u_4$ separately, so we may scale these four points so as to lie at a common height in Minkowski 3-space, say at height one. Since the induced metric is a scalar multiple of the usual metric on this horizontal plane, and since the intersection with the light cone is a round circle in the induced metric, Ptolemy's classical result implies our ideal hyperbolic version.

For our next formula, let us observe that the Möbius group action on Minkowski space is through (Euclidean) volume-preserving linear mappings $SO^+(2,1)$ Thus, the volume of four points in $L^+$ is a well-defined invariant of such a four-tuple, and our next result calculates this invariant, which will play a fundamental role in the sequel.

**Corollary 4.10.** *Suppose $u_1, u_2, u_3, u_4 \in L^+$, where $\bar{u}_1, \bar{u}_2, \bar{u}_3, \bar{u}_4 \in S^1_\infty$ are distinct and occur in this counter-clockwise cyclic order, and let $\lambda_{jk} = \lambda(u_j, u_k) = \sqrt{- < u_j, u_k >}$ for $j, k = 1, 2, 3, 4$, denote the lambda lengths. Then the signed volume of the tetrahedron determined by these four points is given by*

$$2\sqrt{2}\ \lambda_{12}\lambda_{23}\lambda_{34}\lambda_{14}\ \big(\frac{\lambda_{12}^2 + \lambda_{23}^2 - \lambda_{13}^2}{\lambda_{12}\lambda_{23}\lambda_{13}}\ +\ \frac{\lambda_{34}^2 + \lambda_{14}^2 - \lambda_{13}^2}{\lambda_{34}\lambda_{14}\lambda_{13}}\big),$$

*where the volume is positive if and only if the Euclidean segment connecting $u_1, u_3$ lies below the Euclidean segment connecting $u_2, u_4$.*

*Proof.* As usual, we normalize by putting the points into standard position and rely on the basic calculation. Thus, we have the four points in $L^+$ given in the usual coordinates on $\mathbb{R}^3$ as:

$$\begin{aligned}
\frac{ae}{b}u &= \frac{1}{\sqrt{2}}\frac{ae}{b}(-1,0,1),\\
\frac{be}{a}v &= \frac{1}{\sqrt{2}}\frac{be}{a}(1,0,1),\\
\frac{ab}{e}w &= \sqrt{2}\ \frac{ab}{e}(0,-1,1),\\
\zeta &= \frac{1}{\sqrt{2}}\Big( (\frac{d}{ae} - \frac{c}{be})(ac+bd), \frac{2cd}{e}, (\frac{d}{ae} + \frac{c}{be})(ac+bd) - \frac{2cd}{e}\Big),
\end{aligned}$$

and subtracting $\frac{ab}{e}w$ from the others, we therefore have the three corresponding Euclidean displacement vectors

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\,(\frac{(bd-ac)(bd+ac)}{abe}, \frac{2(ab+cd)}{e}, \frac{(ac+bd)^2}{abe} - \frac{2(ab+cd)}{e}),\\
&\frac{1}{\sqrt{2}}\,(-\frac{ae}{b}, \frac{2ab}{e}, \frac{ae}{b} - \frac{2ab}{e}),\\
&\frac{1}{\sqrt{2}}\,(\frac{be}{a}, \frac{2ab}{e}, \frac{be}{a} - \frac{2ab}{e}).
\end{aligned}$$

Now take the triple scalar product of these three vectors in this order and pull out the obvious common factors to get:

$$\frac{\sqrt{2}}{a^2b^2e^3}\begin{vmatrix} (bd-ac)(bd+ac) & ab(ab+cd) & (ac+bd)^2 - 2ab(ab+cd) \\ -ae^2 & ab^2 & ae^2 - 2ab^2 \\ be^2 & a^2b & be^2 - 2a^2b \end{vmatrix}$$

Adding twice the second column plus the first column to the third column, we find:

$$\frac{\sqrt{2}}{a^2b^2e^3}\begin{vmatrix} (bd-ac)(bd+ac) & ab(ab+cd) & 2bd(ac+bd) \\ -ae^2 & ab^2 & 0 \\ be^2 & a^2b & 2be^2 \end{vmatrix}$$

$$= \frac{2\sqrt{2}}{abe^3}\begin{vmatrix} b^2d^2 - a^2c^2 & ab+cd & bd(ac+bd) \\ -ae^2 & b & 0 \\ be^2 & a & be^2 \end{vmatrix},$$

and now directly taking the determinant by expanding along the third column gives

$$\begin{aligned}
&\frac{2\sqrt{2}}{ae}[[ae^2(ab+cd) + b(b^2d^2 - a^2c^2) - a^2d(ac+bd) - b^2d(ac+bd)]\\
&\qquad = \frac{2\sqrt{2}}{e}[e^2(ab+cd) - bac^2 - ad(ac+bd) - b^2cd]\\
&\qquad = 2\sqrt{2}\;abcd\;[\frac{a^2+b^2-e^2}{abe} + \frac{c^2+d^2-e^2}{cde}]
\end{aligned}$$

as was claimed. □

It is worth emphasizing that the expression $\frac{\lambda_{12}^2+\lambda_{23}^2-\lambda_{13}^2}{\lambda_{12}\lambda_{23}\lambda_{13}}$ that occurs in our volume calculation in Corollary 4.10 had already arisen in Lemma 4.6 in our calculation of equidistant points, and it is furthermore worth emphasizing that this same expression

$$\frac{\lambda_{12}^2+\lambda_{23}^2-\lambda_{13}^2}{\lambda_{12}\lambda_{23}\lambda_{13}}=\frac{\lambda_{12}}{\lambda_{23}\lambda_{13}}+\frac{\lambda_{23}}{\lambda_{12}\lambda_{13}}-\frac{\lambda_{13}}{\lambda_{12}\lambda_{23}}$$

is actually linear in the h-lengths by Lemma 4.4. We are aware of no *a priori* geometric or other reason for these "coincidences", and we shall exploit them in subsequent discussions.

One further point we wish to make is that there is actually another simple expression for the volume:

**Corollary 4.11.** *In the notation of Corollary 4.10, the signed volume is also expressed as*

$$2\sqrt{2}\,\big[\lambda_{24}(\lambda_{12}\lambda_{14}+\lambda_{23}\lambda_{34})-\lambda_{13}(\lambda_{12}\lambda_{23}+\lambda_{14}\lambda_{34})\big]$$

*Proof.* This is an algebraic consequence of Corollaries 4.9a and 4.10 since

$$\begin{aligned}
&\frac{a^2+b^2-e^2}{abe}+\frac{c^2+d^2-e^2}{cde}\\
&\qquad=\frac{1}{abcde}\,\big[cd(a^2+b^2-e^2)+ab(c^2+d^2-e^2)\big]\\
&\qquad=\frac{1}{abcde}\,\big[(ac+bd)(ad+bc)-e^2(ab+cd)\big]\\
&\qquad=\frac{1}{abcde}\,\big[ef(ad+bc)-e^2(ab+cd)\big]\\
&\qquad=\frac{1}{abcd}\big[f(ad+bc)-e(ab+cd)\big]
\end{aligned}$$

in the usual notation of the basic calculation, where $f=\frac{ac+bd}{e}$ is the lambda length of the horocycles corresponding to $\zeta$ and $\frac{ab}{e}w$ according to Ptolemy's equation. □

There is one more basic formula from [29] to give at this point, which will not be required in the sequel and is included just because it is perhaps interesting.

**Lemma 4.12.** *For any $q$ in Minkowski 3-space, the function*

$$f_q(p)=\frac{<q-p,q>}{1+z},\ \text{where } p=(x,y,z)\in\mathbb{H},$$

*satisfies the differential equation "of the conformal factor":*

$$f_q^2 \bigtriangleup \log f_q \;=\; -<q,q>(1+<q,q>),$$

*where $\bigtriangleup$ denotes the Laplacian.*

*Proof.* Adopting the usual polar coordinates $r \exp i\theta$ on $\mathbb{D}$, the mapping $\mathbb{D} \to \mathbb{H}$ takes the form

$$r \exp i\theta \mapsto \frac{1}{1-r^2}(2r\cos\,\theta, 2r\sin\,\theta, 1+r^2)$$

in standard Euclidean coordinates on Minkowski space, so in the standard light-cone basis, the image $p \in \mathbb{H}$ is expressed as

$$\begin{aligned}\frac{1+r^2+2r(\sin\,\theta - \cos\,\theta)}{\sqrt{2}(1-r^2)}\;u \;+\; &\frac{1+r^2+2r(\sin\,\theta + \cos\,\theta)}{\sqrt{2}(1-r^2)}\;v\\ -\; &\frac{2r\sin\,\theta}{\sqrt{2}(1-r^2)}\;w.\end{aligned}$$

Setting $q = \alpha u + \beta v + \gamma w$, we find that

$$\begin{aligned} f_q(p) \;&= r^2\big[\alpha+\beta+2\gamma+2\sqrt{2}(\alpha\beta+\alpha\gamma+\beta\gamma)\big]/2\sqrt{2}\\ &+r\big[4\gamma\,\sin\,\theta + 2(\alpha-\beta)\,\cos\,\theta\big]/2\sqrt{2}\\ &+\big[\alpha+\beta+2\gamma-2\sqrt{2}(\alpha\beta+\alpha\gamma+\beta\gamma)\big]/2\sqrt{2}\\ &= Ar^2+Br+C,\end{aligned}$$

where $A$ and $C$ are independent of $\theta$, so

$$\frac{\partial^2 f_q}{\partial r^2} = 2A,\; \frac{\partial f_q}{\partial r} = 2Ar+B,\; \frac{\partial^2 f_q}{\partial \theta^2} = -Br,$$

and

$$\frac{\partial f_q}{\partial \theta} = r[2\gamma\,\cos\,\theta + (\beta-\alpha)\,\sin\,\theta]/\sqrt{2}.$$

We finally calculate that

$$\begin{aligned}
f_q^2 \triangle \log f_q &= [Ar^2+Br+C]\ \big[2A+\frac{2Ar+B}{r}-\frac{B}{r}\big]\\
&\quad -[2Ar+B]^2-\frac{1}{2r}[2\gamma\cos\theta+(\beta-\alpha)\sin\theta]\\
&= 4A[Ar^2+Br+C]-[4A^2r^2+B^2+4ABr]\\
&\quad -\big[2\gamma^2\cos^2\theta+\frac{(\beta-\alpha)^2}{2}\sin^2\theta+2\gamma(\beta-\alpha)\cos\theta\sin\theta\big]\\
&= 4AC-\frac{1}{2}[(\beta-\alpha)^2+4\gamma^2]\\
&= -4(\alpha\beta+\alpha\gamma+\beta\gamma)^2+2(\alpha\beta+\alpha\gamma+\beta\gamma)\\
&= -<q,q>(1+<q,q>)
\end{aligned}$$

as was claimed. □

**Corollary 4.13.** *If $q \in (\mathbb{H} \cup -\mathbb{H}) \cup (L^+ \cup -L^+)$, then $\log f_q(p)$ is a harmonic function of $p \in \mathbb{H}$. Thus if $q_1, q_2$ both lie in this subspace, then*

$$\log \frac{<q_1-p,q_1>}{<q_2-p,q_2>}$$

*is harmonic in p, and in particular,*

$$\frac{1}{2}\log\frac{<p,q>-1}{<p,q>+1}$$

*is Green's function on $\mathbb{H}$ with pole $q \in \mathbb{H}$.*

## 5. Coordinates on decorated Teichmüller spaces

We shall give in this lecture based on [53, 55, 62] parametrizations of several versions of decorated Teichmüller spaces, and we begin with the basic version discussed in the first lecture for a surface $F = F_g^s$ with $s \geq 1$: A point of the Teichmüller space $T(F)$ is a conjugacy class of discrete and faithful representations $\rho : \pi_1(F) \to PSL_2(\mathbb{R})$ of the fundamental group into the Möbius group so that peripheral elements map to parabolics as we have discussed. The subgroup $\Gamma = \rho(\pi_1(F)) < PSL_2(\mathbb{R})$ acts by isometries on the hyperbolic plane, say in the Poincaré disk model $\mathbb{D}$, and there is thus an induced hyperbolic structure on the surface $F = \mathbb{D}/\Gamma$. Equivalently in the Minkowski model, we identify $\Gamma$ with a subgroup (of the same name)

$\Gamma < SO^+(2,1)$, and the hyperbolic structure on the surface is given by $F = \mathbb{H}/\Gamma$.

A point of the decorated Teichmüller space $\tilde{T}(F)$ includes also the further specification of a positive real number to each puncture, and the geometric interpretation of these further parameters is as follows. Supposing that $x \in S^1_\infty$ is the fixed point of some parabolic transformation in $\Gamma$ acting on $\mathbb{D}$, a horocycle in the hyperbolic plane centered at $x$ projects into the surface $F = \mathbb{D}/\Gamma$ to a *closed* curve in $F$, and we refer to such a curve as a *horocycle in* $F$; we say that the horocycle is *centered* at the puncture of $F$ corresponding to $x$. Since a horocycle in $F$ is closed, it has a well-defined finite hyperbolic length in the hyperbolic structure on $F$, and it is this length that we shall identify with the additional real parameter assigned to the corresponding puncture of $F$. Let us emphasize that though a horocycle is necessarily a closed curve, it is not necessarily a simple curve in $F$ though a "short enough" horocycle (depending upon the group $\Gamma$) is always a simple curve separating the puncture from the rest of the surface:

**Lemma 5.1.** *If a collection of distinct horocycles centered at punctures of a hyperbolic surface each has hyperbolic length less than one, then the horocycles are disjointly embedded.*

*Proof.* Choose one of the punctures, and conjugate the corresponding parabolic transformation to $z \mapsto z+1$ in the upper half-space model $\mathcal{U}$. Since the horocycle about the puncture has hyperbolic length less than one, its lift with center $\infty$ lies at height greater than one. If the corresponding horocycle in the surface were not embedded, then there must be some other lift of it with real center which has Euclidean diameter greater than one, and this is a contradiction. Likewise conjugating the parabolic transformation corresponding to the longest horocycle, we derive a similar contradiction. $\square$

A *decorated hyperbolic structure* on $F = F_g^s$ is the specification of a conjugacy class of group $\Gamma$ as above together with the additional data of an $s$-tuple of horocycles, one about each puncture, and taking hyperbolic lengths of horocycles as coordinates on the fibers of $\tilde{T}(F) \to T(F)$, this gives a geometric interpretation to the points of $\tilde{T}(F)$. As a point of notation, we shall typically regard $\Gamma \in T(F)$ suppressing the fact that $\Gamma$ is only defined up to conjugacy, and we shall likewise let $\tilde{\Gamma} \in \tilde{T}(F)$ denote the specification of underlying group $\Gamma$ together with a decoration.

Given $\tilde{\Gamma} \in T(F)$ and given and *ideal arc*, i.e., the homotopy class of an embedded arc $\alpha$ connecting punctures in $F$, we may define the associated *lambda length* $\lambda(\alpha; \tilde{\Gamma})$ in the natural way as the lambda length of any $\Gamma$-geodesic representative for $\alpha$ in the hyperbolic plane with decoration induced from $\tilde{\Gamma}$. In other words, straighten $\alpha$ to a $\Gamma$-geodesic in $F$, truncate using the horocycles from the decoration, let $\delta$ denote the signed hyperbolic length of this truncated geodesic, and set $\lambda(\alpha; \tilde{\Gamma}) = \sqrt{\exp\,\delta}$. In still other words by Lemma 4.1, a lift of the $\Gamma$-geodesic representative of $\alpha$ to $\mathbb{H}$ is asymptotic to a pair of rays in $L^+$, and there are unique points $u, v$ in these rays corresponding via affine duality in Lemma 2.6 to the decoration, and $\lambda(\alpha; \tilde{\Gamma}) = \sqrt{-<u, v>}$.

Define an *ideal triangulation* $\Delta$ of $F_g^s$ to be (the homotopy class of) a collection of disjointly embedded arcs connecting punctures so that each complementary region is a triangle with its vertices among the punctures. Examples of ideal triangulations are illustrated in Figure 10. Euler characteristic considerations show that there are $6g - 6 + 3s$ ideal arcs in an ideal triangulation of $F_g^s$, and there are $4g - 4 + 2s$ complementary triangles.

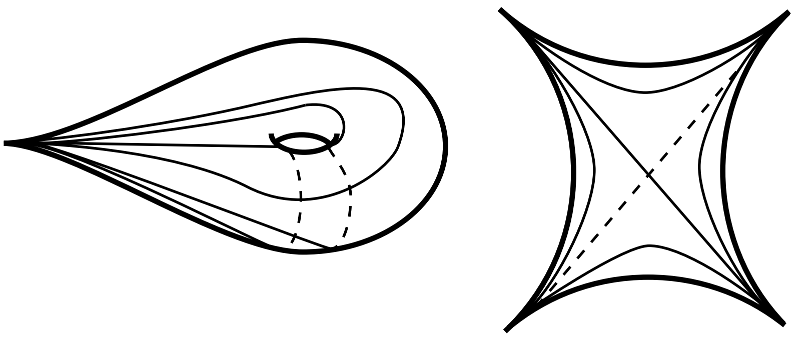


**Figure 10** Examples of ideal triangulations.

**Theorem 5.2.** *For any ideal triangulation $\Delta$ of $F_g^s$ with $s \geq 1$, the natural mapping*

$$\begin{aligned} \Lambda_\Delta : \tilde{T}(F_g^s) &\to \mathbb{R}_{>0}^{\Delta} \\ \tilde{\Gamma} &\mapsto (\alpha \mapsto \lambda(\alpha; \tilde{\Gamma})) \end{aligned}$$

*is a real-analytic surjective homeomorphism.*

*Proof.* We must produce the inverse to $\Lambda_\Delta$ and so suppose there is a positive real number assigned to each arc in $\Delta$. Let $\tilde{F} \to F$ denote the topological universal cover of $F = F_g^s$. The ideal triangulation $\Delta$ lifts to a collection $\tilde{\Delta}$ of arcs decomposing $\tilde{F}$ into triangular regions, and to such each arc is associated the real number assigned to its projection. We shall construct a corresponding collection of decorated geodesics in $\mathbb{D}$, or equivalently, a collection of pairs of points in $L^+$.

Choose one of these topological triangles in $\tilde{F}$, call it $t_0$, and choose an ideal triangle in $\mathbb{D}$, say the triangle $T_0$ spanned by $\bar{u}, \bar{v}, \bar{w}$, where $u, v, w$ is the standard light-cone basis for Minkowski space. The orientation of $\tilde{F}$ induced from that on $F$ gives a cyclic ordering to the vertices of $t_0$ which corresponds to a cyclic ordering of $u, v, w$ as a positively oriented basis for Minkowski space. According to Corollary 4.3, there are unique points in $u, v, w$ realizing the numbers assigned to the edges of $t_0$ as corresponding lambda lengths. This describes a lift of $t_0 \subset \tilde{F}$ to the triangle $\tilde{T}_0 \subset \mathbb{D}$ plus a lift of the ideal points of $t_0$ to a triple of points in $L^+$ covering the vertices of $T_0$, and this completes the basis step of our inductive construction of the inverse to $\Lambda_\Delta$.

For the inductive step, consider one of the three triangular regions $t$ adjacent to $t_0$ in $\tilde{F}$. The common edge $t \cap t_0$ has already been lifted to $\mathbb{D}$ and its vertices to $L^+$ in the basis step. By Lemma 4.7, there is a unique lift $T$ of $t$ to $\mathbb{D}$ and its vertices to $L^+$ that agrees with the lift of $t_0$ so that the lift to $\mathbb{D}$ of the vertex of $t$ disjoint from $t_0$ is separated from $T_0$ by the lift of $t \cap t_0$ and the specified numbers on the edges of $t$ are realized as lambda lengths. We may likewise uniquely lift to $\mathbb{D}$ and $L^+$ the other two triangular regions of $\tilde{F}$ adjacent to $t_0$.

Continue recursively in this way to define a mapping $\phi : \tilde{F} \to \mathbb{D}$ together with lifts of the ideal points of $\tau = \phi(\tilde{\Delta})$ to $L^+$. This mapping $\phi$ is continuous and injective by construction (since we always choose the point in Lemma 4.7 separated from what has previously been constructed), and we claim it is also surjective. To this end, suppose that $z \in \mathbb{D}$ and consider the sequence of triangular regions $t_0, t_1, \ldots$ defined recursively as follows: if $z \notin \phi(t_j)$, then there is a unique triangular region $t_{i+1}$ so that $\phi(t_{i+1})$ and $z$ lie in the same component of $\mathbb{D} - \phi(t_j)$. This sequence of triangles either terminates with $z$ in the image of $\phi$ as desired or else it continues indefinitely.

Passing from triangle $t_j$ to triangle $t_{i+1}$ there are two cases depending upon whether we turn left or right, and there are thus two basic cases for our semi-infinite sequence: either there are infinitely many left and infinitely many right turns in the sequence, or else the sequence ends with an infinite sequence of consecutive left turns or ends with an infinite sequence of consecutive right turns.

In the latter case for instance if the sequence ends with an infinite sequence of common turns $t_k, t_{k+1}, \ldots$, for some $k \geq 0$, then all of the triangles $\phi(t_k), \phi(t_{k+1}), \ldots$ share a common ideal point. Now, there are only finitely many arcs in $\Delta$ and therefore only finitely many possible lambda lengths, hence there are only finitely many possible h-lengths of sectors of triangles complementary to $\tau$ by Lemma 4.4. In particular, these h-lengths are bounded below. Conjugating by the Cayley transform and an appropriate Mobius transformation to guarantee that the common ideal point of the triangles $\phi(t_k), \phi(t_{k+1}), \ldots$ is the point at infinity in $\mathcal{U}$, it follows that the sequence of common turns must terminate, which is a contradiction.

In the former case, there are infinitely many pairs of consecutive triangles $t_k, t_{k+1}$ so that the type, right or left, of the turn $t_{k-1}, t_k$ is different from the type of $t_k, t_{k+1}$. The pair of triangles $\phi(t_k), \phi(t_{k+1})$ determines an ideal quadrilateral, and again since there are only finitely many possible lambda lengths, there are bounds above and below on the cross-ratio of this ideal quadrilateral by Corollary 4.9b. Since the distance between opposite sides of an ideal quadrilateral of bounded cross ratio is itself bounded below, it follow that there can be only finitely many such pairs $t_k, t_{k+1}$, which is again a contradiction.

The mapping $\phi : \tilde{F} \to \mathbb{D}$ is therefore a continuous bijection and indeed a homeomorphism.

Let us now define a homomorphism $\rho : \pi_1(F) \to PSL_2(\mathbb{R})$ which leaves invariant the collection $\tau$ of geodesics. Choose any fundamental domain $D$ for the action of $\pi_1(F)$ on $\tilde{F}$ which is a union of triangular regions complementary to $\tilde{\Delta}$. The action of $\pi_1(F)$ identifies various pairs of frontier edges of $D$, and we may consider such a pair of edges $e_0$ and $e_1$ in the frontier of $D$, say with $\gamma(e_1) = e_0$ for $\gamma \in \pi_1(F)$. There is a unique triangle $t_j$ complementary to $\tilde{\Delta}$ with $e_j$ in its frontier, for $j = 1, 2$, so that $t_0 \subset D$ and $t_1 \not\subset D$. There is then a unique Möbius transformation $\rho(\gamma)$ mapping $\phi(t_1)$ to $\phi(t_0)$ and mapping $\phi(e_1)$ to $\phi(e_0)$ by Corollary 2.5. We may define $\rho(\gamma)$ in this manner for each such pairing $\gamma \in \pi_1(F)$ of edges of $D$ to define a homomorphism $\rho : \pi_1(F) \to PSL_2(\mathbb{R})$, where there are no relations to check since $\pi_1(F)$ is a free group for a punctured surface $F$.

It follows by induction that the representing group $\Gamma = \rho(\pi_1(F)) < PSL_2(\mathbb{R})$ leaves invariant the collection $\tau$ of geodesics by the uniqueness statement in Lemma 4.7. Thus, the conjugacy class of representation $\rho$ is independent of the choice of fundamental domain $D$, and $\phi : \tilde{F} \to \mathbb{D}$ is equivariant for the actions of $\pi_1(F)$ on $\tilde{F}$ and $\Gamma$ on $\mathbb{D}$. Furthermore, the representing group $\Gamma$ is discrete since a sequence of Möbius transformations accumulating at the identity could not leave invariant $\tau$, the representation $\rho$ is faithful since $\phi$ is injective, and $\rho$ maps peripherals to parabolics by construction.

Thus, $\Gamma$ indeed determines an element of $T(F)$, and it remains only to observe that the construction furthermore determines a $\Gamma$-invariant collection of points in $L^+$ lying over the parabolic fixed points of $\Gamma$ in $S^1_\infty$, which descends to a decoration on the hyperbolic surface.

This construction provides a two-sided inverse to the function $\Lambda_\Delta$. That $\Lambda_\Delta$ and its inverse are real-analytic follows from the fact that matrices in $PSL_2(\mathbb{R})$ representing covering transformations can be explicitly computed real analytically in terms of lambda lengths as was described above in the construction of $\rho$ from a choice of fundamental domain. □

Several remarks are in order. One might think of the specification of ideal triangulation $\Delta$ of $F$ as a kind of choice of "basis" for these lambda length coordinates. It is also worth saying explicitly that the basis step of this inductive proof, i.e., the choice of triangular region in the universal cover and the choice of ideal triangle spanned by $\bar{u}, \bar{v}, \bar{w}$, corresponds to normalizing to "kill" the quotient by conjugacy in the definition of Teichmüller space.

We hope that the reader, much as the author, comes away from this result with a firm understanding of what is Teichmüller space: fixing a pattern of gluing triangles to get the specified topological surface $F$, the Teichmüller space of $F$ corresponds to all possible consistent ways of gluing ideal triangles in the specified pattern. The consistency conditions on the gluings arise from the requirement that the resulting metric be complete and can be understood by considering the local picture of the glued triangles near a puncture. Namely, choose a point on an oriented ideal arc incident on a puncture and traverse the horocycle centered at the puncture passing through this chosen point in a given triangle to determine a specified point on the next consecutive arc incident on this puncture; this point, in turn, determines another horocycle in the next triangle and hence a specified point on the next consecutive arc, and so on. After a finite number of such steps, we return to the initial oriented ideal arc, and the point so determined may or may not agree with the initial choice of point on this arc depending on the nature of the gluings. One can see without difficulty that these points agree if and only if the resulting metric is complete near the puncture. Thus, the triangles *cannot* be glued together willy-nilly: there is one consistency condition for each puncture imposed by completeness of the resulting metric. It is worth emphasizing that a real convenience of *decorated* Teichmüller theory is that there are no such consistency conditions on lambda lengths as we have just seen. We shall explicate these conditions on (undecorated) Teichmüller space in Theorem 6.1.

**Lemma 5.3.** *Lambda lengths are natural for the action of the mapping class group, i.e., if $\phi \in MC(F_g^s)$, and $\phi_*(\tilde{\Gamma})$ denotes the push-forward of metric and decoration, then*

$$\lambda(e;\tilde{\Gamma}) = \lambda(\phi(e);\phi_*(\tilde{\Gamma})),$$

*for any ideal arc $e$.*

*Proof.* This follows directly from the definition of lambda lengths, which are invariant under Möbius transformations. □

**Corollary 5.4.** *Suppose that $\Delta$ is an ideal triangulation of $F_g^s$ and $\Lambda$ is an assignment of lambda lengths to the ideal arcs in $\Delta$. If a mapping class on $F_g^s$ leaves $\Delta$ invariant and preserves $\Lambda$, then this mapping class is an isometry of the corresponding hyperbolic surface.*

*Proof.* Manipulating the formula in the previous lemma for some mapping class $\phi$, we have $\lambda(e;\phi_*(\tilde{\Gamma})) = \lambda(\phi^{-1}(e);\tilde{\Gamma})$. Thus, if $\phi$ preserves $\Delta$ and respects lambda lengths, then the coordinates of $\phi_*(\tilde{\Gamma})$ agree with those of $\tilde{\Gamma}$, i.e., $\phi_*$ acts by (decorated) hyperbolic isometry. □

As was mentioned already, the proof of Theorem 5.2 in particular gives an effective algorithm for calculating "holonomies", i.e., matrices representing $\rho(\gamma)$ for $\gamma \in \pi_1(F)$, in terms of lambda lengths (see the proof of Theorem 6.2 for an elegant algorithm to this end), and there is the following special case of particular interest. Recall from Lemma 3.2 that the modular group $PSL_2(\mathbb{Z})$ leaves invariant the Farey tesselation, so any subgroup of the modular group also leaves invariant the Farey tesselation. In particular, if $\Gamma < PSL_2(\mathbb{Z})$ is a finite-index subgroup without elliptic elements, then the Farey tesselation descends to an ideal triangulation on the surface $\mathbb{D}/\Gamma$, and *from the very definition* of the Farey tesselation using horocycles, this surface admits a decoration with all lambda lengths equal to one. Since a "punctured arithmetic surface" corresponds precisely to a finite-index subgroup of the modular group without elliptics, we have:

**Corollary 5.5.** *The collection of punctured arithmetic surfaces corresponds to the set of all ideal triangulations with lambda lengths identically equal to one. Furthermore, the topological symmetry group of the ideal triangulation is the hyperbolic isometry group of the corresponding surface.*

Consider an ideal arc $e$ in an ideal triangulation $\Delta$ of $F$, where we assume that $e$ separates two distinct ideal triangles of $F - \cup\Delta$. In this case, $e$ is a one diagonal of an ideal quadrilateral complementary to $(F - \cup\Delta) \cup e$, and we may replace $e$ by the other diagonal $f$ of this quadrilateral to produce another ideal triangulation $\Delta_e = \Delta \cup \{f\} - \{e\}$ of $F$ as in Figure 11. We say that $\Delta_e$ arises from $\Delta$ by a *flip* along $e$. By Ptolemy's relation Lemma 4.9a, the lambda lengths are related by $ef = ac + bd$, where $a, c$ and $b, d$ are the opposite sides of the quadrilateral.

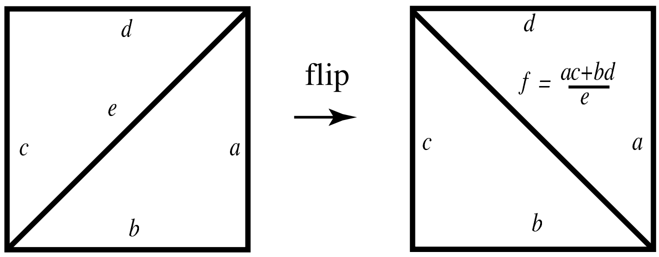



**Figure 11** Flips on ideal traingulations.

**Classical Fact** [Whitehead] Finite sequences of flips act transitively on ideal triangulations of a fixed surface.

See [14] for a proof of this. We shall give a different proof of this fact in Lecture 13 as the first in a heirarchy of such results. Indeed, we shall describe all relations among sequences of flips, relations among relations, and so on.

**Theorem 5.6.** *For any surface $F = F_g^s$ with $s \geq 1$, the action of $MC(F)$ on lambda lengths with respect to a fixed ideal triangulation is described by permutation followed by finite compositions of Ptolemy transformations.*

*Proof.* As before, naturality of lambda lengths gives $\lambda(e; \phi_*(\tilde{\Gamma})) = \lambda(\phi^{-1}(e); \tilde{\Gamma})$ for any $\phi \in PMC(F)$, ideal arc $e$, and $\tilde{\Gamma} \in \tilde{T}(F)$. Given an ideal triangulation, we may thus consider the ideal triangulation $\phi^{-1}(\Delta)$ and simply pull-back the lambda length of $e \in \Delta$ to $\phi^{-1}(e)$ to assign lambda lengths on the ideal arcs in $\phi^{-1}(\Delta)$ which describe $\phi_*(\tilde{\Gamma})$. By the Classical Fact of Whitehead, there is a finite sequence of flips beginning with $\phi^{-1}(\Delta)$ and ending with $\Delta$, and since the effect of a flip on lambda lengths is given by a Ptolemy transformation by Corollary 4.9a, the result follows. □

Fix some ideal triangulation $\Delta$ of $F = F_g^s$ and define a two-form in coordinates with respect to $\Delta$ by

$$\omega_\Delta = \sum d\log\, a \wedge d\log\, b + d\log\, b \wedge d\log\, c + d\log\, c \wedge d\log\, a,$$

where the sum is over all triangles complementary to $\Delta$ in $F$ whose edges have lambda lengths $a, b, c$ in this clockwise cyclic ordering as determined by the orientation of $F$.

**Proposition 5.7.** *This two-form is well-defined independent of ideal triangulation, i.e., if $\Delta, \Delta'$ are two ideal triangulations of $F = F_g^s$, then $\omega_\Delta = \omega_{\Delta'}$ as two-forms on $\tilde{T}(F)$.*

*Proof.* In light of Whitehead's Classical Fact, it suffices to prove the result in case $\Delta' = \Delta_e$, i.e., $\Delta, \Delta'$ differ by a single flip along an ideal arc $e \in \Delta$. To this end, suppose that $e$ separates distinct triangles in $(F - \cup\Delta) \cup e$ with frontier edges $a, b, e$ and $c, d, e$ in these counter-clockwise cyclic orders and let $f$ denote the other diagonal of the quadrilateral with frontier arcs $a, b, c, d$. Adopt the convenient notation that we identify the lambda length of an arc with the arc itself as usual, and set $\tilde{x} = d\log\, x = \frac{dx}{x}$ for $x = a, \ldots, f$, writing the wedge product simply as a product. In particular, by the Ptolemy relation $ef = ac + bd$, we have

$$\tilde{e} + \tilde{f} \;=\; \frac{1}{ac + bd}\big[ac(\tilde{a} + \tilde{c}) \;+\; bd(\tilde{b} + \tilde{d})\big].$$

We may compute the relevant contribution to $\omega_{\Delta'}$ to be

$$\begin{aligned}
&\tilde f\tilde c \,+\, \tilde c\tilde b \,+\, \tilde b\tilde f \,+\, \tilde f\tilde a \,+\, \tilde a\tilde d \,+\, \tilde d\tilde f\\
&= \tilde a\tilde d \,+\, \tilde c\tilde b \,+\, \tilde f(\tilde a \,+\, \tilde c \,-\, \tilde b \,-\, \tilde d)\\
&= \tilde a\tilde d \,+\, \tilde c\tilde b \,-\, \tilde e(\tilde a \,+\, \tilde c \,-\, \tilde b \,-\, \tilde d)\\
&\quad - \frac{ac}{ac+bd}\,(\tilde a \,+\, \tilde c)(\tilde b \,+\, \tilde d) \,+\, \frac{bd}{ac+bd}\,(\tilde b \,+\, \tilde d)(\tilde a \,+\, \tilde c)\\
&= \tilde a\tilde d \,+\, \tilde c\tilde b - \tilde e(\tilde a \,+\, \tilde c \,-\, \tilde b \,-\, \tilde d) \,+\, (\tilde b \,+\, \tilde d)(\tilde a \,+\, \tilde c)\\
&= \tilde e\tilde b \,+\, \tilde b\tilde a \,+\, \tilde a\tilde e \,+\, \tilde e\tilde d \,+\, \tilde d\tilde c \,+\, \tilde c\tilde e,
\end{aligned}$$

which thus agrees with the corresponding contribution to $\omega_\Delta$. □

In fact, this two-form is the pull-back to $\tilde T(F)$ of half the "Weil-Petersson Kähler two-form" on $T(F)$ (cf. Appendix A of [55]); it follows from general facts that it is therefore invariant under $MC(F)$, but as we have just seen, it enjoys the more general invariance under flips. Since the Weil-Petersson two-form is non-degenerate on Teichmüller or moduli space, it furthermore follows that the tangent vectors to the fibers of the forgetful map $\tilde T(F) \to T(F)$ span the vector subspace of the tangent space to $\tilde T(F)$ whose contractions with our pull-back two-form vanish; in other words, scaling all of the lambda lengths meeting a given puncture (where if an arc has both its endpoints at the puncture, then we scale by the square) leaves invariant this two-form, and the corresponding $s$-many vector fields span the degeneracies of the two-form. A direct proof of this fact is described in Remark 6.8.

There are two other versions of the results of this lecture we wish to describe, as follows.

**Partially decorated surfaces** Consider a surface $F = F_g^s$ with $s \geq 1$ as before, and choose among the punctures of $F$ a distinguished non-empty set $P$. A *partial decoration of $F$ on $P$* is the specification of one horocycle centered at each puncture in $P$. Define the *$P$-decorated Teichmüller space* $\tilde T_P(F)$ to be the trivial bundle over $T(F)$ where the fiber over a point is the space of all tuples of horocycles, one horocycle centered at each puncture in $P$.

**Bordered surfaces** Consider a surface $F = F_{g,r}^s$ with boundary components $\partial_1, \ldots, \partial_r$, where $r \geq 1$, and choose on each $\partial_j$ a non-empty set $D_j \subset \partial_j$ of distinguished points. By way of notation, if $D_j$ consists of $\delta_j \geq 1$ points, for $j = 1\ldots, r$, then we define the vector $\vec{\delta} = (\delta_1, \ldots, \delta_r)$ and let $F^s_{g,\vec{\delta}}$ denote a fixed smooth surface $F$ with this extra data. Let us remove from $F$ the distinguished points $D = D_1 \cup \cdots \cup D_r$ on the boundary and double the resulting surface along its boundary arcs so that each point of $D$ gives rise to a puncture of the doubled surface $F'$. A *decoration* on $F^s_{g,\vec{\delta}}$ is a partial decoration of $F'$ on the set $P$ of punctures arising from the distinguished points on the boundary of $F$. There is the natural involution $\iota : F' \to F'$ interchanging the two copies of $F$, and we define the *Teichmüller space* $T(F^s_{g,\vec{\delta}})$ of $F^s_{g,\vec{\delta}}$ to be the $\iota$-invariant subspace of $T(F')$. The *decorated Teichmüller space* $\tilde{T}(F^s_{g,\vec{\delta}})$ of $F^s_{g,\vec{\delta}}$ is defined to be the $\iota$-invariant subspace of $\tilde{T}_P(F')$, i.e., the trivial bundle over $T(F^s_{g,\vec{\delta}})$ where the fiber over a point is the space of all tuples of horocycles, one horocycle about each puncture in $P$.

We begin with the former version and suppose that $P$ is a non-empty subset of the collection of punctures of $F = F^s_g$. Define a *quasi triangulation based at* $P$ to be the homotopy class of a collection of disjointly embedded ideal arcs so that each complementary region is either a triangle or a once-punctured monogon with its vertices in $P$. There are $6g-6+2s+\#P$ ideal arcs in a quasi triangulation based at $P$, $s-\#P$ complementary once-punctured monogons, and $4g-4+s+\#P$ complementary triangles.

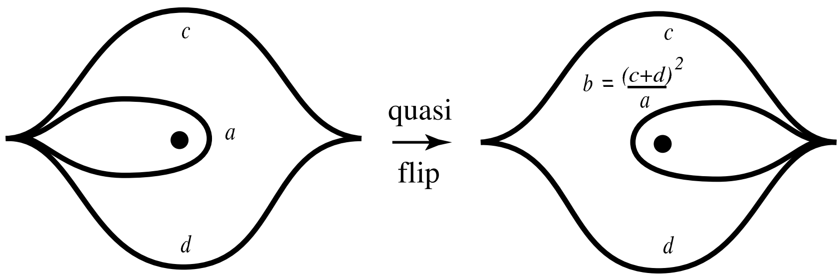



**Figure 12** Quasi flips on quasi triangulations.

It is not true that flips act transitively on quasi triangulations of a fixed surface, however, flips together with an additional combinatorial move do act transitively (cf. Lecture 13). The additional move is called a *quasi flip* and is defined as follows: If an ideal arc $a$ in a quasi triangulation $\Delta$ based at $P$ decomposes a bordered subsurface $F^1_{0,(2)}$ into a triangle and a once-punctured monogon, then the quasi flip along $a$ removes and replaces $a$ with the unique ideal arc $b \neq a$

in $F^1_{0,(2)}$ so that $b$ also decomposes $F^1_{0,(2)}$ into a triangle and a once-punctured monogon as illustrated in Figure 12; again, we denote the resulting quasi triangulation $\Delta_a = \Delta \cup \{b\} - \{a\}$. It is a small exercise using Ptolemy's equation to verify that if $F^1_{0,(2)}$ has boundary arcs $c, d$, then $ab = (c+d)^2$, and we shall refer to this as a *quasi Ptolemy transformation.*

Given $\tilde{\Gamma} \in \tilde{T}_P(F)$ and any ideal arc $e$ connecting points of $P$, there is again a lambda length $\lambda(e;\tilde{\Gamma})$ defined as before, and we have the following result summarizing the material of this section for partially decorated surfaces:

**Theorem 5.8.** *For any quasi triangulation $\Delta$ of $F^s_g$ based at $P$, the natural mapping*

$$\begin{aligned}\Lambda_\Delta : \tilde{T}_P(F^s_g) &\to \mathbb{R}^\Delta_{>0} \\ \tilde{\Gamma} &\mapsto (e \mapsto \lambda(e;\tilde{\Gamma}))\end{aligned}$$

*is a real-analytic surjective homeomorphism. Furthermore, the action of $PMC(F)$ on lambda lengths with respect to a fixed quasi triangulation is described by permutation followed by finite compositions of Ptolemy transformations and quasi Ptolemy transformations. Finally, the two-form $\omega_\Delta$ defined exactly as before, summing only over triangles complementary to $\Delta$ in $F$, is well-defined independent of the quasi triangulation $\Delta$.*

*Proof.* We may extend the quasi triangulation to an ideal triangulation by adding the unique ideal arc from undecorated puncture to decorated puncture in each complementary once-punctured monogon, and we may assign any lambda length to each of these added ideal arcs. Follow through the proof of Theorem 5.2 *verbatim* and notice that by the last sentence in Lemma 4.7, the resulting representation of the fundamental group in the Möbius group is independent of the choices of lambda lengths on the added ideal arcs. The proof that the action of $PMC(F)$ is as stated follows exactly as before (and note that we could actually have taken instead the subgroup of $MC(F)$ that fixed the distinguished punctures $P$ setwise). To see that the two-form is invariant, adopt the notation in the definition of quasi Ptolemy transformation, where

$ab = (c+d)^2$, and compute as in the proof of Proposition 5.7:

$$\begin{aligned}\tilde{d}\tilde{c} + \tilde{c}(\frac{c+d}{a})^{\sim} &= \tilde{d}\tilde{c} - \tilde{c}\tilde{a} - \tilde{a}\tilde{d} + 2(\tilde{c} - \tilde{d})\frac{c\tilde{c} + d\tilde{d}}{c+d} \\ &= \tilde{d}\tilde{c} + \tilde{a}\tilde{c} + \tilde{d}\tilde{a} + 2\tilde{c}\tilde{d} \\ &= \tilde{c}\tilde{d} + \tilde{a}\tilde{c} + \tilde{d}\tilde{a},\end{aligned}$$

as desired. □

A case of particular interest, which we shall further discuss later (cf. Theorem 6.1), is when $P$ is a singleton, i.e., among the punctures, we have chosen a distinguished one (so in particular for $F_g^1$, there is no choice). In this case, let us note that changing the partial decoration simply scales each lambda length by a common amount; indeed, moving the horocycle a hyperbolic distance $d$ changes the hyperbolic length along geodesics by the amount $2d$, and hence scales the lambda lengths by an amount $\exp d$. This gives:

**Corollary 5.9.** *For any quasi triangulation $\Delta$ of $F_g^s$ based at a single puncture, projective classes of lambda lengths of ideal arcs in $\Delta$ give a real-analytic parametrization of Teichmüller space $T(F_g^s)$, the action of the mapping class group is given by permutation together with Ptolemy and quasi Ptolemy transformations, and the two-form described before is well-defined independent of quasi triangulation.*

Let us turn finally to a bordered surface $F = F^s_{g,(\delta_1,\ldots,\delta_r)}$ with distinguished points $D$ in its boundary, with double $F'$ and with involution $\iota : F' \to F'$ defined as before. The arcs in the boundary of $F$ connecting consecutive points of $D$ give rise to a family of ideal arcs in $F'$ which we denote by $B$, where $\#B = \#D = \delta_1 + \cdots + \delta_r$, and the distinguished points in the boundary of $F$ give rise to a family $P$ of punctures of $F'$.

A *quasi triangulation* $\Delta$ of $F$ is the restriction to $F$ of any $\iota$-invariant quasi triangulation of $F'$ based at $P$, so a quasi-triangulation of $F$ automatically contains $B$. There are thus $6g-6+3r+2s+2(\delta_1+\cdots+\delta_r)$ ideal arcs in a quasi triangulation of $F$. We may perform flips or quasi flips on ideal arcs in $\Delta - B$, and we may imagine performing the corresponding $\iota$-equivariant flips or quasi flips in $F'$. Again, we shall prove (cf. Lecture 13) that flips and quasi flips on ideal arcs in $\Delta - B$ act transitively on quasi triangulations of $F$ and have:

**Theorem 5.10.** *For any quasi triangulation $\Delta$ of $F^s_{g,\vec{\delta}}$, the natural mapping*

$$\Lambda_\Delta : \tilde{T}(F^s_{g,\vec{\delta}}) \to \mathbb{R}^\Delta_{>0}$$
$$\tilde{\Gamma} \mapsto (e \mapsto \lambda(e; \tilde{\Gamma}))$$

*is a real-analytic surjective homeomorphism. Furthermore, the action of $PMC(F)$ on lambda lengths with respect to a fixed quasi triangulation is described by permutation followed by finite compositions of Ptolemy transformations and quasi Ptolemy transformations. Finally, the two form $\omega_\Delta$ defined exactly as before, summing only over triangles complementary to $\Delta$ in $F$, is well-defined independent of the quasi triangulation $\Delta$.*

*Proof.* In light of our definitions, this truly follows directly from Theorem 5.8 and Whitehead's Classical Fact. □

Each of these variants, partially decorated surfaces and bordered surfaces, will find non-trivial applications in the sequel. Avoiding the temptation to go overboard, let us simply observe that there is a still more elaborate variant of the theory where one considers bordered surfaces together with a distinguished subset of the punctures and decorates not only the distinguished points on the boundary but also these distinguished punctures.

## 6. Coordinates on Teichmüller spaces

As an alternative to the parametrization of $T(F^s_g)$ given by projectivized lambda lengths on a quasi triangulation based at a single puncture given in Corollay 5.9, we shall in this lecture discuss using instead cross ratios of adjacent pairs of triangles complementary to an ideal triangulation. To this end, suppose that points $\xi_j \in S^1_\infty$, for $j = 1, 2, 3, 4$, are the vertices at infinity in this counter-clockwise order of an ideal quadrilateral in $\mathbb{D}$, and triangulate this quadrilateral by the diagonal connecting $\xi_1$ to $\xi_3$. We may conjugate by a conformal map in the two distinct ways: 1) sending $\xi_1 \mapsto 0$, $\xi_2 \mapsto 1$, $\xi_3 \mapsto \infty$, which maps $\xi_4$ to some negative real number $-\zeta_1$; and 2) sending $\xi_3 \mapsto 0$, $\xi_4 \mapsto 1$, $\xi_1 \mapsto \infty$, which maps $\xi_2$ to some negative real number $-\zeta_2$. It follows from Lemma 4.9c that $Z = \log \zeta_1 = \log \zeta_2$ is the signed hyperbolic distance between the orthogonal projections of $\xi_2$ and $\xi_4$ to the specified diagonal of the quadrilateral. Furthermore, choosing any decoration on the quadrilateral, say with points $u_j \in L^+$ covering $\xi_j$, for $j = 1, 2, 3, 4$, Lemma 4.9b gives

$$\begin{aligned} Z &= \frac{1}{2} \log \frac{< u_2, u_3 > < u_1, u_4 >}{< u_1, u_2 > < u_3, u_4 >} \\ &= \log \frac{\lambda(h(u_2), h(u_3))\ \lambda(h(u_1), h(u_4))}{\lambda(h(u_1), h(u_2))\ \lambda(h(u_3), h(u_4))}. \end{aligned}$$

We shall call $Z$ the *shear coordinate* associated to the triangulated quadrilateral, and it is evidently independent of an ordering on the vertices of the quadrilateral as well as independent of decoration though it does depend upon a choice of diagonal triangulating the quadrilateral. Likewise given a once-punctured monogon triangulated by an edge $e$ connecting the puncture to the distinguished point on the boundary, consider the lifts of this region to the universal cover of the surface; the edge $e$ lifts to the diagonal of a quadrilateral, and the corresponding shear coordinate is found to vanish again by Lemma 4.9b. Thus in any case, to an arc $e$ in an ideal triangulation $\Delta$ of $F_g^s$ with specified hyperbolic structure $\Gamma \in T(F_g^s)$, there is an associated shear coordinate $Z_\Delta(e; \Gamma)$, which vanishes if $e$ does not separate distinct triangles complementary to $\Delta$.

**Theorem 6.1.** *For any ideal triangulation $\Delta$ of $F_g^s$ with $s \geq 1$, the natural mapping*

$$\begin{aligned} T(F_g^s) &\to \mathbb{R}^\Delta \\ \Gamma &\mapsto (e \mapsto Z_\Delta(e; \Gamma)) \end{aligned}$$

*is a real-analytic homeomorphism onto the linear subspace determined by the following equations: for each puncture $p$ of $F_g^s$, we have*

$$\sum Z_\Delta(e; \Gamma) = 0,$$

*where the sum is over all arcs $e$ of $\Delta$ which are asymptotic to $p$ counted with multiplicity, i.e., if $e$ has both endpoints at $p$, then $Z_\Delta(e; \Gamma)$ occurs twice in the sum.*

*Proof.* The argument is entirely analogous to the proof of Theorem 5.2, where we lift $\Delta$ to an ideal triangulation of the topological universal cover $\tilde{F}$ of $F = F_g^s$, choose a triangular region complementary to $\tilde{\Delta}$ in $\tilde{F}$ and an ideal triangle in $\mathbb{D}$ to begin the recursive construction of the mapping $\phi : \tilde{F} \to \mathbb{D}$ determined by the putative shear coordinates, which are again invariant under Möbius transformations. The inductive step of the recursion depends upon the fact that the cross ratio is a complete invariant of ordered four-tuples of points in $S^1_\infty$. This mapping $\phi$ is again a continuous injection by construction. In the earlier proof that $\phi$ was surjective, there were two cases for a semi-infinite sequence of turns: either there were infinitely many left-followed-by-right and right-followed-by-left turns or the sequence ended with a consecutive semi-infinite sequence of left or right turns. The former case is again impossible since there are only finitely many distinct shear coordinates. To rule out the latter case, consider uniformizing in upper half-space $\mathcal{U}$ so that the consecutive triangles all share the point $\infty$ at infinity as a common vertex. Letting $x_j \in \mathbb{R}$, for $j \geq 0$, denote the consecutive further vertices of these triangles, we find that the sequential shear coordinates are given by $\pm\log \frac{x_j - x_{j-1}}{x_{j+1}-x_j}$, so this sequence might indeed be bounded above and below with the sequence $x_j$ accumulating to some finite value. However, this expression for shear coordinates shows that the constraint on these coordinates telescopes, and there is some $n \geq 1$ so that for each $k \geq 1$, we have $x_{kn+1} - x_{kn} = x_1 - x_0$. This possibility is thus untenable, and the mapping $\phi : \tilde{F} \to \mathbb{D}$ is again a surjective homeomorphism. The induced tesselation of $\mathbb{D}$ is invariant by a subgroup $\Gamma < PSL_2(\mathbb{R})$ defined in analogy to Theorem 5.2 again using that the cross ratio is a complete invariant of ordered four-tuples of points in $S^1_\infty$.

The necessity of the asserted constraints on shear coordinates can be checked directly using the formula Lemma 4.9b relative to any decoration on $F$. Alternatively, using the interpretation of shear coordinates in Lemma 4.9c as distances between orthogonal projections likewise establishes necessity of these constraints; this also proves sufficiency of the constraints since peripheral elements represented in the constructed group $\Gamma$ only then preserve horocycles and hence are parabolic Möbius transformations. □

Returning to the discussion following Theorem 5.2, we see that a gluing of ideal triangles produces a complete hyperbolic structure on the punctured surface precisely when the constraints of Theorem 6.1 hold. Again, it is worth emphasizing that passing from Teichmüller space to decorated Teichmüller space, these constraints disappear, and lambda lengths give global unconstrained coordinates.

Since the linear constraints on shear coordinates were used only to guarantee that the constructed mapping $\phi : \tilde{F} \to \mathbb{D}$ was surjective and that peripheral elements were represented by parabolics, this suggests that removing these constraints may correspond to dropping this restriction on peripheral elements. Indeed this is the case as we next discuss.

Fix some surface $F = F_g^s$ with $s \geq 1$ and consider the space

$$Hom'' \;=\; Hom''(\pi_1(F_g^s), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})$$

of conjugacy classes of discrete and faithful representations $\rho$ of $\pi_1(F)$ in $PSL_2(\mathbb{R})$ so that if $\gamma \in \pi_1(F)$ is a peripheral element, then the absolute value of the trace of $\rho(\gamma)$ is at least two. A peripheral element $\gamma \in \pi_1(F)$ is therefore represented by either a parabolic or a hyperbolic element. In this context, we shall refer to the punctures of $F_g^s$ as "holes", where a hole is a "$\rho$-puncture" if the trace has absolute value two and is a "$\rho$-boundary" if the trace has absolute value greater than two for $\rho \in Hom''$, where these attributes are actually associated with holes rather than just peripheral elements by invariance of trace under conjugacy.

We define a $2^s$-fold branched cover

$$\Pi : \hat{T}(F) \to Hom'',$$

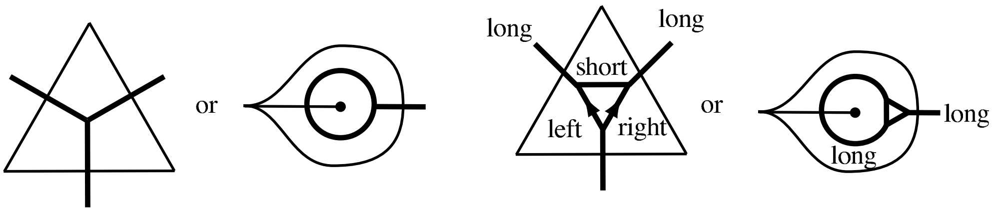
where the fiber over a point of $Hom''$ is the collection of all tuples of orientations on the components of the $\rho$-boundary. In particular, $T(F_g^s)$ is identified with a subspace of $\hat{T}(F)$, and if $r_1 + s_1 = s$, then there are $2^{r_1}$ canonical embeddings of $T(F_{g,r_1}^{s_1})$ into $\hat{T}(F)$ corresponding to the possible orientations.

In order to associate a shear coordinate to a point $\hat{\Gamma} \in \hat{T}(F)$ and an arc $e$ in $F$ connecting holes, we may remove a small neighborhood of the holes of $F$ that are $\Pi(\hat{\Gamma})$-boundary components in order to consider $e$ as an arc in a surface with punctures and boundary. Associated with the point $\hat{\Gamma}$, there is furthermore an orientation associated with each $\Pi(\hat{\Gamma})$-boundary component, and we may "spin" $e$ around each boundary component in the sense determined by its orientation. More precisely, each geodesic boundary component of the surface $\mathbb{D}/\Pi(\hat{\Gamma})$ lifts to a geodesic in $\mathbb{D}$ that lies in the frontier of the universal cover $\tilde{F} \subset \mathbb{D}$, and we may define a lift of $e$ to $\tilde{F}$ by sliding its endpoint along this geodesic to infinity in the sense determined by the specified orientation, finally straightening to a geodesic for $\Pi(\hat{\Gamma})$. Thus, the specification of $\hat{\Gamma} \in \hat{T}(F)$ and an ideal triangulation $\Delta$ of $F_g^s$ gives a well-defined collection of arcs indexed by $\Delta$ decomposing $F$ into ideal triangles, and each such arc $e \in \Delta$ has a well-defined shear coordinate $Z_\Delta(e; \hat{\Gamma})$ defined as before.

**Theorem 6.2.** [Thurston-Fock] *For any ideal triangulation $\Delta$ of the surface $F = F_g^s$ with $s \geq 1$, the natural mapping*

$$\hat{T}(F) \to \mathbb{R}^\Delta$$
$$\hat{\Gamma} \mapsto (e \mapsto Z_\Delta(e; \hat{\Gamma}))$$

*is a real-analytic homeomorphism onto. Furthermore, the hyperbolic length of a $\rho$-boundary component is given by the absolute value of the sum of incident shear coordinates counted with multiplicity.*




**13a** Fatgraph from triangulation. **13b** Freeway from fatgraph.

**Figure 13** Fatgraphs and freeways.

Before giving the proof, we must first prepare some background combinatorial topology from [54], and in fact, it is convenient to use a combinatorial formalism different from ideal triangulations in the current discussion. Namely, we shall describe "cubic fatgraph spines" in a surface $F = F_g^s$, which are defined as follows. Suppose that $\Delta$ is an ideal triangulation of $F$. Construct the "Poincaré dual" $G = G(\Delta)$ of $\Delta$ in $F$ by which we mean: there is one vertex of $G$ for each region complementary to $\Delta$, and for each arc of $\Delta$ separating two (not necessarily distinct) triangles, there is a "dual" edge of $G$ connecting the corresponding vertices; see Figure 13a. Thus, each vertex of $G$ has valence three, and we say simply that $G$ is "cubic". The graph $G$ is a "spine" of $F$ in the sense that there is a strong deformation retraction of $F$ onto $G$, and furthermore, $G$ comes equipped with an additional structure, namely, the orientation of $F$ induces a clockwise ordering on the arcs in the frontier of a complementary region to $\Delta$, and hence there is a corresponding cyclic ordering on the dual half-edges of $G$ about each vertex. This latter structure of such a cyclic ordering about each vertex is called a "fattening" on the graph $G$, and a graph with this extra structure is called a "fatgraph".

Notice that an arc $e$ in $\Delta$ separates distinct triangles if and only if its dual edge in $G$ has distinct endpoints. In this case, the flip $\Delta \to \Delta_e$ is defined, and the dual fatgraphs $G(\Delta)$ and $G(\Delta_e)$ are related by a

*Whitehead move*, i.e., collapsing and expanding the edge of $G(\Delta)$ dual to $e$ in the natural way as in Figure 14 produces the fatgraph $G(\Delta_e)$. We shall more generally and systematically study fatgraphs later (in Lecture 11).

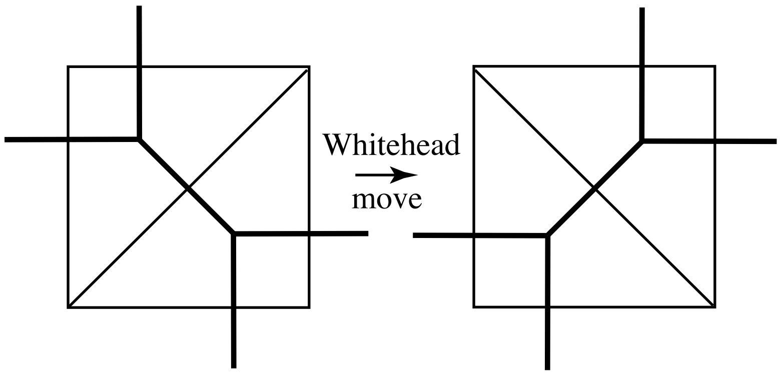



**Figure 14** Whitehead moves on fatgraphs.

Following [52], we furthermore modify the cubic fatgraph $G = G(\Delta)$ to produce the corresponding "freeway" $G'$ by replacing each vertex of $G$ by a corresponding triangular region in $G'$ as illustrated in Figure 13b. We shall refer to a frontier edge of the added triangular regions as "short" edges of $G'$ and to the other edges, which are in one-to-one correspondence with the edges of $G$ itself, as the "long" edges of $G'$. An orientation of a short edge either "turns right", i.e., agrees with the counter-clockwise orientation on the boundary of a small triangular region, or it "turns left", i.e., disagrees with this orientation, as is also illustrated in Figure 13b.

*Proof.* We must again provide an inverse to the mapping defined by taking shear coordinates and shall accomplish this by directly constructing a corresponding class of representations in $Hom''$ from the putative coordinates.

As in the earlier discussion, suppose that $Z$ is the shear coordinate of the triangulated quadrilateral in $\mathbb{D}$ with ideal vertices corresponding to $0, 1, \infty$ and $-e^Z$ triangulated by the geodesic connecting $0$ to $\infty$. Consider the Möbius transformation

$$X_Z = \begin{pmatrix} 0 & -e^{\frac{Z}{2}} \\ e^{-\frac{Z}{2}} & 0 \end{pmatrix}.$$

One sees directly that $X_Z$ interchanges 0 with $\infty$ and 1 with $-e^Z$, and indeed $X_Z^2 = 1 \in PSL_2(\mathbb{R})$; thus, $X_Z$ describes the unique elliptic Möbius transformation fixing the geodesic asymptotic to $0, \infty$ and mapping $1 \mapsto -e^Z$, which also happens to map $-e^Z \mapsto 1$ and hence is an involution. There are two further elliptic Möbius transformations of immediate interest, namely,

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } L = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix},$$

and again one sees directly that $R$ maps $0 \mapsto \infty \mapsto -1 \mapsto 0$, $L$ maps $0 \mapsto -1 \mapsto \infty \mapsto 0$, so each of $R, L$ are order three elements of $PSL_2(\mathbb{Z})$ with $L = R^{-1}$.

We next directly define a representation $\rho : \pi_1(F) \to PSL_2(\mathbb{R})$ given putative shear coordinates on the ideal triangulation $\Delta$. We may regard the shear coordinates as associated to the long edges of the freeway $G'$ derived from the cubic fatgraph $G = G(\Delta)$ dual to $\Delta$, and we choose as basepoint for the fundamental group of $F$ a vertex $*$ of $G'$. Since $G$ is a spine of $F$, any closed curve $\gamma$ in $F$ based at $*$ is homotopic to a closed edge-path in $G'$ starting at $*$. Such an edge-path is uniquely described by a corresponding sequence $e_1, \ldots, e_{n+1}$ of consecutive oriented edges of $G'$, and we define

$$\rho(\gamma) = M(e_{n+1}) \cdots M(e_1) \in PSL_2(\mathbb{R}),$$

where

$$M(e) = \begin{cases} X_Z, & \text{if } e \text{ is long and has shear coordinate } Z; \\ R, & \text{if } e \text{ is short and turns right;} \\ L, & \text{if } e \text{ is short and turns left.} \end{cases}$$

Insofar as $R^3 = L^3 = RL^{-1} = X_Z^2 = 1 \in PSL_2(\mathbb{R})$, it follows that $\rho$ is well-defined and indeed a representation of $\pi_1(F)$. We must prove that $\rho$ is discrete, faithful, determines orientations on the $\rho$-boundary components, and realizes the putative shear coordinates.

To these ends, consider a hole of $F$ so that the sum of coordinates on incident edges is non-zero, where the coordinates are counted with multiplicity as before, and remove a topological open disk neighborhood to produce a corresponding boundary component. Performing this modification for each such puncture of $F$, we produce a surface $F'$ with boundary. Restrict the ideal triangulation $\Delta$ of $F$ to a collection of arcs in $F'$, and spin these arcs around each such boundary component, spinning to the left if the sum of coordinates is negative and to the right if the sum of coordinates is positive. This produces a finite family $\Delta'$ of arcs decomposing $F'$ into triangular regions and determines an orientation on each boundary component of $F'$, where the orientation agrees with the direction of this spinning.

As in the proof of Theorem 5.2, consider the topological universal cover $\tilde{F}'$ of $F'$ and the lift $\tilde{\Delta}'$ of $\Delta'$ to $\tilde{F}'$. We may choose a triangular region complementary to $\tilde{\Delta}'$ in $\tilde{F}'$ and an ideal triangle in $\mathbb{D}$ to begin the recursive construction of the mapping $\phi : \tilde{F}' \to \mathbb{D}$ determined by the putative shear coordinates as before. Again by construction, $\phi$ is a continuous injection which conjugates the action of $\gamma \in \pi_1(F)$ on $\tilde{F}$ to the action of $\rho(\gamma)$ on $\phi(\tilde{F}')$, and furthermore, $\rho(\pi_1(F)) < PSL_2(\mathbb{R})$ leaves invariant the collection $\phi(\tilde{\Delta}')$ of geodesics in $\mathbb{D}$.

We claim that each boundary component of $F'$ is a $\rho$-boundary. To see this for a fixed such boundary component, we may change base point and conjugate by a Möbius transformation so that the first edge of the corresponding edge-path is a small edge of the freeway lying in the triangle chosen in $\tilde{F}'$ to begin the recursive construction of $\phi$. As in the proof of Theorem 6.1, we may uniformize in upper half-space $\mathcal{U}$ so that the geodesic connecting $x_0 = 0$ to $\infty$ lies in the frontier of the ideal triangle chosen in $\mathbb{D}$ to begin the recursive construction of $\phi$. There are two cases depending upon the orientation on the fixed boundary component. Suppose first that the arcs in $\Delta'$ twist to the left along this boundary component of $F'$, so the consecutive lifts to $\mathcal{U}$ of these arcs have ideal points $\infty$ and $x_0 < x_1 < x_2 < \cdots$. Let $Z_j$ denote the shear coordinate of the arc connecting $\infty$ to $x_j$, so we have as before that $Z_j = \log \frac{x_j - x_{j-1}}{x_{j+1} - x_j}$ for $j \geq 1$. Let $n$ denote the number of arcs spinning around this hole again counted with multiplicity, so that for all $k \geq 0$, we have $Z_{kn+j} = Z_j$. It follows from a geometric sequence that $x_j$ converges to $x_\infty = x_n(1 - e^{-z})^{-1}$ as $j$ tends to infinity, where $z = \sum_{j=1}^n Z_j$ and $z < 0$ by our construction. Furthermore on the based curve going once around this hole in its specified orientation, $\rho$ takes value

$$\begin{aligned}(X_{Z_n}L) \cdots (X_{Z_1}L) &= \begin{pmatrix} e^{\frac{Z_n}{2}} & e^{\frac{Z_n}{2}} \\ 0 & e^{-\frac{Z_n}{2}} \end{pmatrix} \cdots \begin{pmatrix} e^{\frac{Z_1}{2}} & e^{\frac{Z_1}{2}} \\ 0 & e^{-\frac{Z_1}{2}} \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{z}{2}} & -2x_\infty \sinh \frac{z}{2} \\ 0 & e^{-\frac{z}{2}} \end{pmatrix},\end{aligned}$$

where the final off-diagonal entry follows from the fact that $x_\infty$ is invariant under this Möbius transformation as the limit of the invariant sequence $x_1, x_2, \ldots$.

The argument for the case that the arcs twist to the right is entirely analogous except that $\sum_{j=1}^n Z_j > 0$ and we replace the matrix $L$ by $R$ in the earlier calculation. This completes the proof that each boundary component of $F'$ is in fact a $\rho$-boundary, and indeed, that the $\rho$-length of this boundary component is given by $|\sum_{j=1}^{n} Z_j|$ in either case since the absolute value of the trace of a Möbius transformation is twice the hyperbolic cosine of half its translation length along the invariant geodesic.

It remains only to prove that the group $\Gamma = \rho(\pi_1(F))$ is discrete. To see this, we may choose a connected fundamental domain for $\Gamma$ consisting of a finite collection of ideal triangles since $\phi(\Delta')$ is invariant under $\Gamma$ by construction. It follows that each element of $\Gamma$ must map each such ideal triangle to a disjoint ideal triangle, and discreteness follows. $\square$

It is worth remarking explicitly that the previous proof gives an elegant description of the holonomy $\rho(\gamma)$ for each $\gamma \in \pi_1(F)$ in terms of the representing edge-path on any corresponding freeway. Indeed, though Theorem 6.2 was well-known among the Thurston school in the 1980's, cf. [67], this aspect of the proof we have presented was Fock's more recent innovation [15], and it is the basis for the quantization [13, 35] of $\hat{T}(F)$, where the shear coordinates are replaced by appropriate operators on a Hilbert space. The action of flips on shear coordinates is calculated using lambda lengths in Theorem 6.3. The other key ingredient for quantization beyond the combinatorial description of the mapping class group given in Lecture 13 is the Poission structure on $\hat{T}(F)$ (i.e., a skew-symmetric pairing on smooth functions on $\hat{T}(F)$ that satisfies the Jacobi and Leibnitz identities) corresponding to the Weil-Petersson Kähler two-form described in Proposition 5.7, and this Poisson structure is described in Theorem 6.5 and its center in Theorem 6.7.

**Theorem 6.3.** *Suppose that $\Delta$ is an ideal triangulation of $F$ and $\hat{\Gamma} \in \hat{T}(F)$. Suppose that $e \in \Delta$ separates two triangles complementary to $\Delta$, where these two triangles have frontier arcs $a, b, e \in \Delta$ and $c, d, e \in \Delta$ in these correct clockwise orders. Perform a flip on $e$ to produce the ideal triangulation $\Delta' = \Delta_e$, let $e'$ denote the unique arc in $\Delta' - \Delta$, set $X = Z_\Delta(x; \hat{\Gamma})$ and $X' = Z_{\Delta'}(x; \hat{\Gamma})$, for $x = a, \cdots, d$, set $E = Z_\Delta(e; \hat{\Gamma})$, $E' = Z_{\Delta'}(e'; \hat{\Gamma})$, and define $\Phi(X) = \log(1 + e^X)$. Provided $a, b, c, d$ are all distinct, then $E' = -E$ and*

$$\begin{aligned} A' &= A + \Phi(E), \quad B' = B - \Phi(-E), \\ C' &= C + \Phi(E), \quad D' = D - \Phi(-E). \end{aligned}$$

*In the following special cases, these formulas are to be modified as follows:*

$$a = c \;\Rightarrow\; A' = A + 2\Phi(E);\;\; b = d \;\Rightarrow\; B' = B - 2\Phi(-E);$$
$$a \in \{b, d\} \;\Rightarrow\; A' = A + E;\;\; c \in \{b, d\} \;\Rightarrow\; C' = C + E.$$

*Proof.* Begin with the case that $a, b, c, d$ are all distinct and adopt the notation that arcs bounding a triangle together with $x$ other than $a, b, e$ and $c, d, e$ are $x_1, x_2, x$ in this clockwise cyclic order, for $x = a, b, c, d$. It is easiest to simply calculate in lambda lengths for some fixed decoration, where we identify an arc with its lambda length for simplicity. Thus, $E = \log \frac{ac}{bd}$,

$$\begin{aligned} A &= \log \frac{a_1 b}{a_2 e},\; B = \log \frac{b_1 e}{b_2 a}, \\ C &= \log \frac{c_1 d}{c_2 e},\; D = \log \frac{d_1 e}{d_2 c}, \end{aligned}$$

so $E' = \log \frac{bd}{ac} = -E$, and

$$\begin{aligned} A' &= \log \frac{a_1 e'}{a_2 d} = \log \frac{a_1 b}{a_2 e} \frac{ac + bd}{bd} = A + \log\,(1 + \frac{ac}{bd}), \\ B' &= \log \frac{b_1 c}{b_2 e'} = \log \frac{b_1 e}{b_2 a} \frac{ac}{ac + bd} = B - \log\,(1 + \frac{bd}{ac}), \\ C' &= \log \frac{c_1 e'}{c_2 b} = \log \frac{c_1 d}{c_2 e} \frac{ac + bd}{bd} = C + \log\,(1 + \frac{ac}{bd}), \\ D' &= \log \frac{d_1 a}{d_2 e'} = \log \frac{d_1 e}{d_2 c} \frac{ac}{ac + bd} = D - \log\,(1 + \frac{bd}{ac}), \end{aligned}$$

using Ptolemy's equation $ee' = ac + bd$, as was claimed.

In the special cases, we likewise compute:

$$\begin{aligned} a = c \;\Rightarrow\; A' &= \log \frac{e'^2}{bd} = \log \frac{bd}{e^2}\,(\frac{ac + bd}{bd})^2 = A + 2\Phi(E); \\ b = d \;\Rightarrow\; B' &= \log \frac{ac}{e'^2} = \log \frac{e^2}{ac}\,(\frac{ac}{ac + bd})^2 = B - 2\Phi(-E); \\ a = b \;\Rightarrow\; A' &= \log \frac{c}{d} = \log \frac{b}{a} \frac{ac}{bd} = A + E; \\ a = d \;\Rightarrow\; A' &= \log \frac{a}{d} = \log \frac{b}{c} \frac{ac}{bd} = A + E; \\ c = b \;\Rightarrow\; C' &= \log \frac{c}{b} = \log \frac{d}{a} \frac{ac}{bd} = C + E; \\ c = d \;\Rightarrow\; C' &= \log \frac{a}{b} = \log \frac{d}{c} \frac{ac}{bd} = C + E; \end{aligned}$$

□

**Corollary 6.4.** *In the notation of Theorem 6.3, consider the subsurface $\bar{F}$ of $F$ comprised of the two triangles with frontier edges $a, b, e$ and $c, d, e$. Let $\alpha$ be the boundary of a regular neighborhood in $\bar{F}$ of one of the vertices of these triangles. Then the sum of shear coordinates of arcs in $\Delta$ meeting $\alpha$ agrees with the sum of the shear coordinates of arcs in $\Delta'$ meeting $\alpha$ counted with multiplicity, i.e., if an arc in $\Delta$ or $\Delta'$ meets $\alpha$ twice, then its shear coordinate contributes twice to the sum.*

*Proof.* In particular, if $a, b, c, d$ are all distinct, then $\bar{F}$ is a quadrilateral embedded in $F$, the assertion is that

$$A' + D' = A + D + E,\ A' + B' + E' = A + B,$$
$$B' + C' = B + C + E,\ C' + D' + E' = C + D,$$

and these relations follow from the formulas in Theorem 6.3 using the identity

$$\Phi(E) - \Phi(-E) \ = \ \log \frac{1 + e^E}{1 + e^{-E}} \ = \ \log e^E \ = \ E.$$

In the special cases, we likewise compute from Theorem 6.3:

if $a = c$ (and similarly if $b = d$), then

$$C' + 2D' + E' = C + 2D + E,\ A' + 2B' + E' = A + 2B + E;$$

if $a = b$ (and similarly if $c = d$), then

$$A' = A + E,\ A' + C' + D' = A + C + D + 2E,\ C' + D' + E' = C + D;$$

if $a = d$ (and similarly if $b = c$), then

$$A' = A + E,\ A' + B' + C' + 2E' = A + B + C,\ B' + C' = B + C + E;$$

if $a = c$ and $b = d$, then

$$A' + B' + E' = A + B + E;$$

if $a = d$ and $b = c$ (and similarly if $a = b$ and $c = d$), then

$$A' = A + E,\ B' = B + E,\ A' + B' + 2E' = A + B.$$

□

Suppose that $\Delta$ is an ideal triangulation of $F = F_g^s$, for $s \geq 1$, with dual cubic fatgraph spine $G$. If $a, b \in \Delta$ are distinct, then let $\epsilon_{ab}$ be the number of components of $F - (\cup\Delta \cup G)$ whose frontier contains points of $a$ and points of $b$ counted with a sign that is positive if $a$ and $b$ are consecutive in the clockwise order (arising from the orientation of $F$) in the corresponding region and with a negative sign if $a$ and $b$ are consecutive in the counter-clockwise order. Setting $\epsilon_{aa}=0$ for each $a\in\Delta$, $\epsilon_{ab}$ takes the possible values $0,\pm 1,\pm 2$ and comprise the entries of a skew-symmetric matrix indexed by $\Delta$. For any $a\in\Delta$, regard the corresponding shear coordinate $Z_a = Z_\Delta(a;\hat\Gamma)$ as a real-valued coordinate function defined on $\hat T(F)$, and define a Poisson structure by setting $\{Z_a, Z_b\}_\Delta=\epsilon_{ab}$, the constant function on $\hat T(F)$ with value $\epsilon_{ab}$, where we extend linearly using the Leibnitz rule to an appropriate class of functions in the shear coordinates on $\hat T(F)$.

The Poission bracket $\{\cdot,\cdot\}$ of two functions depends only upon their differentials, so there is a corresponding skew-symmetric two-tensor $\eta$ called the "Poisson bivector" so that for any two functions $f,g$, we have $< df\otimes dg, \eta >= \{f,g\}$, where $<\cdot,\cdot>$ is induced by the pairing between cotangent and tangent vectors. From the definition of our Poisson structure on $\hat T(F)$, the corresponding Poisson bivector is given by

$$\eta_\Delta=\sum \frac{\partial}{\partial Z_a}\wedge\frac{\partial}{\partial Z_b}+\frac{\partial}{\partial Z_b}\wedge\frac{\partial}{\partial Z_c}+\frac{\partial}{\partial Z_c}\wedge\frac{\partial}{\partial Z_a},$$

where the sum is over all triangles complementary to $\Delta$ in $F$ whose edges in this clockwise cyclic ordering as determined by the orientation of $F$ have shear coordinates $Z_a, Z_b, Z_c$, and this evidently restricts to the Poisson structure induced on $T(F_g^s)\subset \hat T(F)$ by the symplectic two-form $\omega_\Delta$ described in Proposition 5.7 (using also the remarks about degeneracies of this two-form following the same result).

**Theorem 6.5.** *The Poisson structure $\{\cdot,\cdot\}_\Delta$ on $\hat T(F)$ is well-defined independent of the choice of ideal triangulation $\Delta$ of $F=F_g^s$.*

*Proof.* By Whitehead's Classical Fact from the previous lecture that flips act transitively on ideal triangulations, it suffices to show that if one ideal triangulation arises from another by a single flip, then the Poission structures for these two ideal triangulations coincide. In the notation of Theorem 6.3, suppose first that the arcs $a,b,c,d$ are all distinct. We may compute with the bivector and set $\partial_X=\frac{\partial}{\partial X}$ for $X=A,B,C,D,E$ and $\partial'_X=\frac{\partial}{\partial X'}$, for $X=A',B',C',D',E'$, so $\partial_X=\partial'_X$ for $X=A,B,C,D$, and

$$\partial_E \;=\; \frac{e^E}{1+e^E}\left(\partial'_A+\partial'_C\right)\;+\;\frac{e^{-E}}{1+e^{-E}}\left(\partial'_B+\partial'_D\right)\;-\;\partial'_E$$

by Theorem 6.3. The relevant contribution to $\eta_\Delta$ is thus given by

$$\begin{aligned}
&\partial_A \wedge \partial_B + \partial_B \wedge \partial_E + \partial_E \wedge \partial_A + \partial_C \wedge \partial_D + \partial_D \wedge \partial_E + \partial_E \wedge \partial_C \\
&= \partial'_A \wedge \partial'_B + \partial'_C \wedge \partial'_D \\
&\quad + \ (\partial'_B + \partial'_D - \partial'_A - \partial'_C) \\
&\qquad \wedge \ \Big[\frac{e^E}{1+e^E}(\partial'_A + \partial'_C) + \frac{e^{-E}}{1+e^{-E}}(\partial'_B + \partial'_D) - \partial'_E\Big] \\
&= \partial'_A \wedge \partial'_B + \partial'_C \wedge \partial'_D \\
&\qquad + \ \partial'_E \wedge (\partial'_B + \partial'_D - \partial'_A - \partial'_C) \ + \ (\partial'_B + \partial'_D) \wedge (\partial'_A + \partial'_C) \\
&= \partial'_A \wedge \partial'_D + \partial'_D \wedge \partial'_E + \partial'_E \wedge \partial'_A + \partial'_C \wedge \partial'_B + \partial'_B \wedge \partial'_E + \partial'_E \wedge \partial'_C,
\end{aligned}$$

giving the relevant contribution to the bivector $\eta_{\Delta'}$, as required, where we have used the identity $\frac{e^E}{1+e^E} + \frac{e^{-E}}{1+e^{-E}} = 1$ in the second equality.

In the special cases, we always have $\partial'_X = \partial_X$ for $X = A, B, C, D$, and in the various cases, the calculation is performed analogously with:

if $a = c$ (and similarly if $b = d$), then

$$\partial'_E + \partial_E = \frac{2e^E}{1+e^E}\ \partial_A + \frac{e^{-E}}{1+e^{-E}}\ (\partial_B + \partial_D);$$

if $a = b$ (and similarly if $c = d$), then

$$\partial'_E + \partial_E = \partial_A + \frac{e^E}{1+e^E}\ \partial_C + \frac{e^{-E}}{1+e^{-E}}\ \partial_D;$$

if $a = d$ (and similarly if $b = c$), then

$$\partial'_E + \partial_E = \partial_A + \frac{e^{-E}}{1+e^{-E}}\ \partial_B + \frac{e^E}{1+e^E}\ \partial_C;$$

if $a = c$ and $b = d$, then

$$\partial'_E + \partial_E = \frac{2e^E}{1+e^E}\ \partial_A + \frac{2e^{-E}}{1+e^{-E}}\ \partial_B;$$

if $a = b$ and $c = d$ (and similarly if $a = d$ and $b = c$), then

$$\partial'_E + \partial_E = \partial_A + \partial_C.$$

□

**Example 6.6.** For the surface $F_1^1$, there is only one combinatorial type of ideal triangulation, as illustrated in Figure 10. Letting $A, B, C$ denote the three shear coordinates, we have $\{A, B\} = \{B, C\} = \{C, A\} =$

2, and the matrix of Poisson brackets is given by

$$\begin{pmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}.$$

The center of the Poisson algebra is generated by the kernel of this matrix spanned by $2(A+B+C)$ corresponding to the unique hole.

**Theorem 6.7.** *Fix an ideal triangulation $\Delta$ of $F = F_g^s$ and index the holes of $F$ by $p = 1, \ldots, s$. Consider the sum $C_p$ of the shear coordinates of the arcs in $\Delta$ incident on the hole $p$ counted with multiplicity as before. Then the center of the Poisson algebra is freely generated by $\{C_p : p = 1, \ldots, s\}$.*

*Proof.* We begin by proving that each $C_p$ is indeed central and suppose that $e$ is an arbitrary arc in $\Delta$ with $E = Z_\Delta(e; \hat\Gamma)$. There are two cases depending upon whether $e$ triangulates a once-punctured monogon or a quadrilateral complementary to $\Delta - \{e\}$, and in the former case, let $f \in \Delta$ denote the frontier of this once-punctured monogon and set $F = Z_\Delta(f; \hat\Gamma)$. It may be that the hole $p$ corresponds to the puncture in this monogon, in which case $C_p = E$, so of course $\{E, C_p\} = 0$ by skew-symmetry. If $p$ corresponds to another hole of $F$, then we may write $C_p = \mathcal{C} + \delta F$, where $\delta = 0, 1$ and $\{E, \mathcal{C}\} = 0$. Since $\{E, F\} = 0$ by definition of the bracket, we conclude that $\{E, C_p\} = 0$ as required.

In the latter case, let us adopt the notation of Theorem 6.3 for the nearby edges $x = a, b, c, d$ and shear coordinates $X = Z_\Delta(x; \hat\Gamma)$, and suppose first that $a, b, c, d$ are all distinct. Any element $C_p$ is of the form $C_p = \mathcal{C} + \sum_{j=1}^4 \delta_j \mathcal{D}_j$, where $\mathcal{D}_1 = A+B$, $\mathcal{D}_2 = C+D$, $\mathcal{D}_3 = A+D+E$, $\mathcal{D}_4 = B+C+E$, $\{E, \mathcal{C}\} = 0$, and each $\delta_j = 0, 1$. One checks using the definition of the bracket that $\{E, \mathcal{D}_j\} = 0$, for $j = 1, 2, 3, 4$, so indeed $\{E, C_p\} = 0$ in this case.

There are again special cases if $a, b, c, d$ are not distinct, which we may summarize as follows:

if $a = c$ (and similarly if $b = d$), then

$$C_p = \mathcal{C} + \delta_1(A + 2B + E) + \delta_2(A + 2D + E);$$

if $a = b$ (and similarly if $c = d$), then

$$C_p = \mathcal{C} + \delta_1 A + \delta_2(C + D) + \delta_3(A + C + D + 2E);$$

if $a = d$ (and similarly if $b = c$), then

$$C_p = \mathcal{C} + \delta_1(A + E) + \delta_2(A + B + C) + \delta_3(B + C + E);$$

if $a = c$ and $b = d$, then

$$C_p = 2(A + B + E);$$

if $a = b$ and $c = d$, then

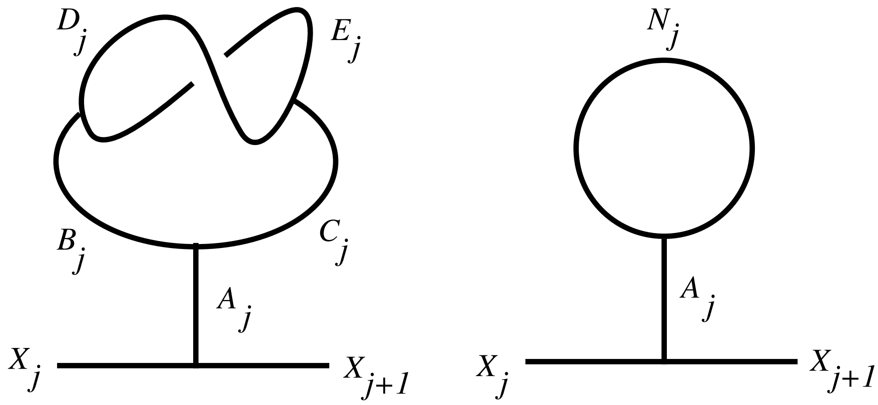
$$C_p = A, C, \text{ or } A + C + 2E$$

if $a = d$ and $b = c$, then

$$C_p = A + B, A + E, \text{ or } B + E,$$

where $\{E, \mathcal{C}\} = 0$, the $\delta$'s are equal to zero or one, and in each case the bracket of $E$ with each possible term again vanishes by definition.

Thus, the asserted elements are indeed central. According to Corollary 6.4, these elements are furthermore invariant under flips, and by Theorem 6.5, the center of the Poisson algebra is likewise invariant under flips. Using the Classical Fact from the previous lecture that flips act transitively on ideal triangulations (cf. Lecture 13), we may thus prove the current result for any particularly convenient ideal triangulation, and it then follows for any ideal triangulation. We shall assume that $2g + 2s > 5$ since the unique surface $F_1^1$ of negative Euler characteristic ruled out by this inequality has already been handled separately in Example 6.6.




**15a** Increase genus **15b** Add a puncture

**Figure 15** Building blocks for convenient fatgraph.

We may equivalently describe the dual fatgraph to this convenient ideal triangulation, and it is comprised of various "building blocks" of the two types depicted in Figure 15 joined together in a line (where the cyclic ordering in the fatgraph structure is inherited from the plane of projection of our figure); for the surface $F_g^s$, there are $g$ copies of the building block in Figure 15a and $s-1$ copies of the building block in Figure 15b. It is clear that the $s$ elements $C_p$ for this ideal triangulation are linearly independent, and we have already shown that they are indeed central. We complete the proof by calculating that the center of the Poisson algebra for the ideal triangulation dual to this cubic fatgraph has rank $s$.

Adopting notation for the shear coordinates $A, B, C, D, E$ in Figure 15a, we have the corresponding matrix of Poisson brackets between these variables given by

$$\begin{array}{c|c|cccc|}
 & A_j & B_j & C_j & D_j & E_j \\
\hline
A_j & 0 & 1 & -1 & 0 & 0 \\
\hline
B_j & -1 & 0 & 1 & 1 & -1 \\
C_j & 1 & -1 & 0 & 1 & -1 \\
D_j & 0 & -1 & -1 & 0 & 2 \\
E_j & 0 & 1 & 1 & -2 & 0 \\
\hline
\end{array} .$$

Add the last row to the next-to-the-last row, add the last column to the next-to-the-last column, add the third row to the second row, and finally add the third column to the second column to obtain

$$\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & -2 \\
1 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 2 \\
0 & 2 & 1 & -2 & 0
\end{pmatrix},$$

which evidently has rank four and can be further reduced (without adding the first column or row to any other) to yield

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +2 \\
0 & 0 & 0 & -2 & 0
\end{pmatrix}.$$

We may thus erase from the matrix of Poisson brackets of all edges those columns and rows corresponding to the variables $B_j$, $C_j$, $D_j$, and $E_j$ without changing the rank of this matrix.

Likewise, adding a building block as in Figure 15b creates exactly one degeneracy since the variable $N_j$ Poisson commutes with all other variables as we have already shown.

It remains only to calculate the rank of the matrix corresponding to a graph with edges $X_j$ and $A_j$ remaining after erasing all $B$-, $C$-, $D$-, $E$-, and $N$-variable rows and columns. The corresponding Poisson bracket matrix has odd dimension $2g+2s-5>0$ and the block-diagonal form

$$\begin{array}{r|rr|rr|rr|r}
0 & 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\
\hline
-1 & 0 & 1 & 0 & 0 & & & \\
1 & -1 & 0 & 1 & -1 & & & \\
\hline
 & 0 & -1 & 0 & 1 & 0 & 0 & \\
 & 0 & 1 & -1 & 0 & 1 & -1 & \\
\hline
 & & & 0 & -1 & 0 & 1 & \\
 & & & 0 & 1 & -1 & 0 & \ddots \\
\hline
 & & & & & & \ddots & \ddots
\end{array}\ .$$

Add each even-index row to its predecessor and add each even-index column to its predecessor to produce a matrix whose only nonzero elements are $+1$ on the main super-diagonal and -1 on the main sub-diagonal. This matrix has rank $2g+2s-6$, which completes the proof. $\square$

*Remark* 6.8. The exactly parallel discussion and *identical* calculations show that tangent vectors to Teichmüller space which contract to zero with the invariant two-form discussed in the previous section are tangent to the fibers of the projection to Teichmüller space as was discussed before, and in particular, these vector fields are linearly independent. The same remark applies in the setting of partially decorated or bordered surfaces. For a bordered surface $F=F^s_{g,(\delta_1,\ldots,\delta_r)}$, however, there is a further variant which is sometimes studied as follows. An ideal triangulation $\Delta$ of $F$ in particular contains the arcs $B$ lying in the boundary of $F$, and we may consider the two-form defined in analogy to the previous section, summing over triangles complementary to $\Delta$ in $F$ as before, but including only summands $d\log x\wedge d\log y$ for pairs of arcs lying in $\Delta-B$, where we have as usual identified an arc with its lambda length for convenience; put another way, we may consider the lambda lengths on the arcs in $B$ as fixed *a priori* and pull-back the two-form considered before. In this case, the degeneracies of this modified two-form are spanned by the following vectors. For each $j=1,\ldots,r$ with $\delta_j$ even, enumerate in their correct cyclic order induced from the orientation on the boundary the distinguished points $p_1,\ldots,p_{\delta_j}$ occurring on the $jth$ boundary component. Enumerate the arcs of $\Delta-B$ incident on $p_k$ by $a^k_\ell$, for $k=1,\ldots,\delta_j$ and $\ell=1,\ldots,L_k$, and form the sums $\lambda_k=\sum_{\ell=1}^{L_k}\frac{\partial}{\partial\log\ a^k_\ell}$, where again this sum is taken with multiplicity (so if $a^k_\ell$ is asymptotic to $p_k$ in both directions, then there are two contributions to the sum). Finally forming the alternating sum

$\Lambda_j = \lambda_1 - \lambda_2 + \cdots - \lambda_{\delta_j}$ for $\delta_j$ even, the degeneracies of the modified two-form are freely spanned by $\{\Lambda_j : \delta_j \text{ is even}\}$. The proof is again exactly parallel to the proof of Theorem 6.7, now using also that an even-dimensional matrix whose non-zero entries are +1 on the main super-diagonal and -1 on the main sub-diagonal is non-singular.

## 7. Circle homeomorphisms

Perhaps surprisingly but as we shall see in the next two lectures, the previous considerations have applications to circle homeomorphisms and harmonic analysis. This lecture is principally based on [56].

A *tesselation* $\tau$ of $\mathbb{D}$ is a countable collection of geodesics decomposing $\mathbb{D}$ into ideal triangles, where $\tau$ is required also to be locally finite, i.e., any point of $\mathbb{D}$ admits a neighborhood meeting only finitely many geodesics in $\tau$. We shall let $\tau^0 \subset S^1_\infty$ denote the collection of ideal points of all the geodesics in $\tau$. The requirement that complementary regions to $\cup\tau$ are ideal triangles implies that $\tau^0$ is dense in $S^1_\infty$. (Note that $\tau$ is *not* locally finite in the closed disk $\mathbb{D} \cup S^1_\infty$.)

A principal example is the Farey tesselation $\tau_*$ which has already been discussed in the third lecture. As we have seen, $\tau^0_*$ is the image of the rational numbers plus infinity under the Cayley transform, and we shall simply identify $\tau^0_*$ with this set denoted $\tau^0_* = \bar{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\} \subset S^1_\infty$. Thus, $\frac{p}{q} \in \bar{\mathbb{Q}}$ is identified with the point $\frac{p+iq}{p-iq} \in S^1_\infty \subset \mathbb{C}$ thereby establishing a bijection between $\tau^0_*$ and the set of all points in the unit circle whose coordinates are rational.

For another example of a tesselation, we might begin with the triangle spanned by $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ and recursively define the ideal triangle adjacent to an edge with endpoints $\xi, \eta \in S^1_\infty$ already constructed to have its other vertex at the Euclidean midpoint of the appropriate circular segment with endpoints $\xi, \eta$. We shall call this the *dyadic tesselation* $\tau_d$ since its set $\tau^0_d$ of ideal vertices is the collection of all points of the form $e^{2\pi i\theta}$, where $\theta = \frac{p}{2^n}$ is a dyadic rational, for some $n \in \mathbb{Z}$ and some odd $p \in \mathbb{Z}$.

More generally, given any countable dense subset $S \subset S^1_\infty$ plus a bijection $\beta : \mathbb{N} \to S$, there is a corresponding tesselation $\tau$ defined as follows. Begin with the ideal triangle spanned by $\beta(1), \beta(2), \beta(3)$ and recursively define the ideal triangle adjacent to an edge with endpoints $\xi, \eta \in S^1_\infty$ already constructed to have its other vertex at $\beta(i)$, where $i \in \mathbb{N}$ is the least index with $\beta(i)$ in the appropriate circular segment with endpoints $\xi, \eta$. Density of $S$ in $S^1_\infty$ shows that $\mathbb{D} - \cup\tau$ consists of ideal triangles with vertices in $S$, and it is easy to see that $\tau$ is locally finite by construction. In fact, any tesselation of $\mathbb{D}$ clearly arises in this way thus giving an explicit sense of the collection of objects we are studying here, namely, enumerated countable dense subsets of the circle.

A *distinguished oriented edge* or *doe* on a tesselation $\tau$ is simply the specification of an orientation on some geodesic in $\tau$. We shall take the edge connecting $\frac{0}{1}$ to $\frac{1}{0}$ as the standard doe on $\tau_*$ and $\tau_d$ letting $\tau_*'$ and $\tau_d'$, respectively, denote the Farey and dyadic tesselations with this choice of doe. Let $\mathcal{T}ess'$ denote the collection of all tesselations with doe of $\mathbb{D}$.

A key point is that tesselations with doe are "combinatorially rigid" in the following sense. Choosing $\tau_*'$ as a kind of basepoint for $\mathcal{T}ess'$ and given any other $\tau' \in \mathcal{T}ess'$, there is a canonically defined mapping $f = f^0_{\tau'} : \tau^0_* \to \tau^0$ defined recursively as follows. Start by defining $f(\frac{0}{1})$ and $f(\frac{1}{0})$, respectively, to be the initial and terminal points of the doe of $\tau'$. There is a unique ideal triangle complementary to $\tau$ lying to the right of the doe, and $f(\frac{1}{1})$ is defined to be the ideal vertex of this triangle distinct from $f(\frac{0}{1})$ and $f(\frac{1}{0})$. This defines $f$ on the vertices of an ideal polygon $P$ with frontier in $\tau_*$.

If $e$ is a geodesic in $\mathbb{D}$ and $f$ is a homeomorphism of $S^1\infty$, then we let $f(e)$ denote the geodesic spanned by its vertex images and similarly let $f(P)$ denote the ideal polygon spanned by the images of vertices of $P$.

We recursively extend $f$ in this same manner as before: a geodesic $e$ in the frontier of $P$ also lies in the frontier of a unique ideal triangle complementary to $\tau_*$ whose interior is disjoint from $P$, there is likewise a unique ideal triangle complementary to $\tau$ containing $f(e)$ in its frontier whose interior is disjoint from $f(P)$, and we extend $f$ by mapping the vertex of the former triangle distinct from the endpoints of $e$ to the vertex of the latter triangle distinct from the endpoints of $f(e)$.

Given $\tau' \in \mathcal{T}ess'$, we have thus defined the mapping $f^0_{\tau'} : \tau^0_* \to \tau'$, and by construction, this mapping is order-preserving. An elementary argument shows that an order-preserving mapping defined on a dense subset of the circle interpolates a unique orientation-preserving homeomorphism of the circle. The homeomorphism of the circle induced in this way by $f^0_{\tau'}$ is denoted

$$f_{\tau'} : S^1_\infty \to S^1_\infty$$

and is called the *characteristic mapping* of $\tau' \in \mathcal{T}ess'$.

Let $Homeo_+(S^1)$ denote the topological group of all orientation-preserving homeomorphisms of $S^1_\infty$ with the compact-open topology, i.e., a sub-basis for this Hausdorff topology is given by those functions that map a specified compact set on $S^1_\infty$ to a specified open set in $S^1_\infty$. The assignment $\tau' \mapsto f_{\tau'}$ gives a mapping of $\mathcal{T}ess'$ into $Homeo_+(S^1)$.

**Lemma 7.1.** *The mapping*

$$\begin{aligned}\mathcal{T}ess' &\to Homeo_+(S^1)\\ \tau' &\mapsto f_{\tau'}\end{aligned}$$

*is a bijection.*

*Proof.* Suppose that $f \in Homeo_+(S^1)$, and define a collection $\tau = \{f(e) : e \in \tau_*\}$, where as before, if $e \in \tau_*$ has ideal points $\xi, \eta$, then $f(e)$ is the geodesic with ideal points $f(\xi), f(\eta)$. $\tau$ is evidently a countable family of geodesics. Density of $\tau_*^0$ and continuity of $f$ imply density of $f(\tau_*^0)$ from which it follows that complementary regions to $\tau$ are indeed ideal triangles, and local finiteness of $\tau$ follows from that of $\tau_*$. Thus, $\tau$ is a tesselation, and we take as doe the oriented geodesic connecting $f(\frac{0}{1})$ to $f(\frac{1}{0})$ to produce $\tau'_f \in \mathcal{T}ess'$. The assignment $f \to \tau'_f$ is a two-sided inverse to $\tau' \mapsto f_{\tau'}$. □

We induce a topology on $\mathcal{T}ess'$ using the previous lemma. The Möbius group $PSL_2(\mathbb{R})$ acts continuously on the left on $Homeo_+(S^1)$ by composition, and there is an induced topology on the quotient. Furthermore, $f \in Homeo_+(S^1)$ also acts on $\tau' \in \mathcal{T}ess'$ in the natural way, where $f(\tau) = \{f(e) : e \in \tau\}$ in the previous notation, with the doe on $f(\tau)$ given by the image under $f$ of the doe for $\tau$. The assignment of characteristic mapping is evidently equivariant for these actions. Define the quotient space

$$\mathcal{T}ess = \mathcal{T}ess'/PSL_2(\mathbb{R}),$$

which is thus homeomorphic to $Homeo_+(S^1)/PSL_2(\mathbb{R})$ and of principal interest.

We say that $\tau' \in \mathcal{T}ess'$ is *normalized* if the doe connects the point $\frac{0}{1}$ to the point $\frac{1}{0}$ and if the other vertex of the triangle to the right of the doe is the point $\frac{1}{1}$. Likewise, we say that $f \in Homeo_+(S^1)$ is *normalized* if $f$ fixes each of the points $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$. Since a Möbius transformation is determined by its values at three points, we conclude that $\mathcal{T}ess$ is homeomorphic to the collection of all normalized tesselations with doe and that $Homeo_+(S^1)/PSL_2(\mathbb{R})$ is homeomorphic to the collection of all normalized orientation-preserving homeomorphisms of the circle. In particular, $\mathcal{T}ess$ and $Homeo_+(S^1)/PSL_2(\mathbb{R})$ are infinite-dimensional

Hausdorff spaces. In effect, we introduced doe's only to kill them by taking the quotient by the Möbius group, and this little manipulation was performed in order to produce a space of orbits which is Hausdorff.

Given a tesselation $\tau'$ with doe representing a point of $\mathcal{T}ess$, there is a well-defined shear coordinate assigned to each geodesic in $\tau$ defined just as in the previous lecture. We may use the characteristic mapping $f_{\tau'}$ to index these shear coordinates by edges of $\tau_*$ in the natural way, i.e., given $e \in \tau_*$, the geodesic $f(e) \in \tau$ triangulates an ideal quadrilateral complementary to $\tau - \{e\}$, and the logarithm of the cross ratio of this quadrilateral computed as before is called the *shear coordinate* of $\tau'$ on $e$.

**Theorem 7.2.** *The assignment of shear coordinates gives an embedding*

$$\mathcal{T}ess \approx Homeo_+(S^1)/PSL_2(\mathbb{R}) \to \mathbb{R}^{\tau_*}$$

*onto an open subspace, where the target is given the weak topology, thus endowing $\mathcal{T}ess$ with the structure of an infinite-dimensional Fréchet space.*

*Proof.* Since the cross ratio is a complete invariant of a four-tuple of points under the action of the Möbius group, the assigment of shear coordinates is indeed an injection as before. Continuity of this mapping follows from the definition of the topology on $\mathcal{T}ess$ as induced by the compact-open topology on $Homeo_+(S^1)$ and the definition of the weak topology on a function space, where a sub-basis is given by allowing only finitely many coordinates to vary. $\square$

Myriad interesting questions and problems arise: Choose your favorite class of homeomorphisms of the circle (smooth, Hölder of some exponent, really any class), and ask for a topological characterization of the corresponding tesselations or for the characterization in terms of shear coordinates. In particular, characterize the image of this embedding. Describe the inverse of a circle homeomorphism in terms of tesselations or in coordinates. Describe composition of circle homeomorphisms in terms of tesselations or coordinates. Are inversion and composition Fréchet maps in this structure? Among the first class of questions, we presently know just one answer, which we shall present in Theorem 7.6.

A *decoration* on a tesselation $\tau$ is the assignment of one horocycle centered at each point of $\tau^0$. The space $\widetilde{\mathcal{T}ess}'$ of decorated tesselations with doe admits a natural topology as a $\mathbb{R}^{\omega}_{>0}$-bundle over $\mathcal{T}ess'$, where the fiber is given the weak topology. $\widetilde{\mathcal{T}ess}'$ again admits a natural continuous left action of the Möbius group acting not only on tesselation with doe as before but also on horocycles, and we define the quotient

$$\widetilde{\mathcal{T}ess} = \widetilde{\mathcal{T}ess}'/PSL_2(\mathbb{R}).$$

By definition, the map $\widetilde{\mathcal{T}ess} \to \mathcal{T}ess$ which forgets decoration is a continuous surjection.

Given a decorated tesselation $\tilde{\tau}'$ with doe representing a point of $\widetilde{\mathcal{T}ess}$, there is a well-defined lambda length assigned to each geodesic in $\tau$ defined just as in the classical case: the decoration on the triangle to the right of the doe determines lambda lengths on the frontier edges of this triangle by Lemma 4.3, and lambda lengths on consecutive edges are then uniquely determined according to Lemma 4.7. We may again use the characteristic mapping $f_{\tau'}$ to index these lambda lengths by edges of $\tau_*$ in the natural way.

**Theorem 7.3.** *The assignment of lambda lengths gives an embedding*

$$\widetilde{\mathcal{T}ess} \to \mathbb{R}^{\tau_*}_{>0}$$

*onto an open subspace, where the target is given the weak topology, thus endowing $\widetilde{\mathcal{T}ess}$ with the structure of an infinite-dimensional Fréchet space.*

*Proof.* The proof closely parallels the classical case, where the lambda lengths are used to uniquely determine a mapping $\tau_*^0 \to L^+$, the only significant point being continuity of the mapping, which is guaranteed by our definition of topologies. $\square$

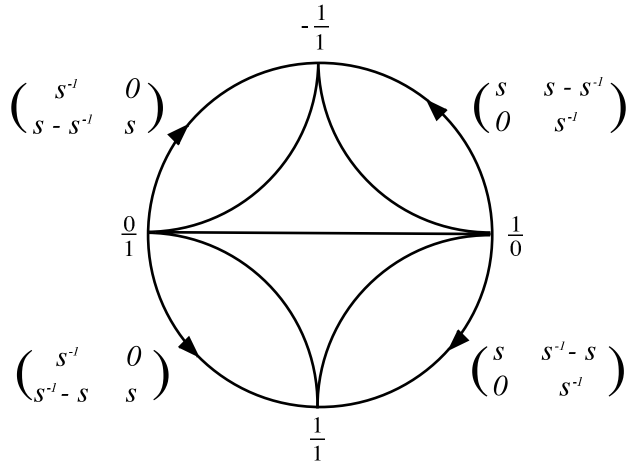



**Figure 16** The multiplicative group $\Lambda(s)$.

There is an associated basic deformation of lambda lengths described as follows. The four points $\frac{0}{1}, \frac{1}{0}, \pm\frac{1}{1} \in S^1_\infty$ decompose $S^1_\infty$ into four component arcs, one in each of the quadrants $I, II, III, IV$ in $\mathbb{R}^2$ (enumerated in counter-clockwise order starting from quadrant $I$ where both coordinates are positive). For $s \in \mathbb{R}_{>0}$, define a piecewise-Möbius mapping on $S^1_\infty$ by

$$\Lambda(s)(\xi) = \begin{cases} \begin{pmatrix} s & s - s^{-1} \\ 0 & s^{-1} \end{pmatrix} \xi, & \text{for } \xi \text{ in quadrant } I; \\ \begin{pmatrix} s^{-1} & 0 \\ s - s^{-1} & s \end{pmatrix} \xi, & \text{for } \xi \text{ in quadrant } II; \\ \begin{pmatrix} s^{-1} & 0 \\ s^{-1} - s & s \end{pmatrix} \xi, & \text{for } \xi \text{ in quadrant } III; \\ \begin{pmatrix} s & s^{-1} - s \\ 0 & s^{-1} \end{pmatrix} \xi, & \text{for } \xi \text{ in quadrant } IV. \end{cases}$$

See Figure 16. The four points $\frac{0}{1}, \frac{1}{0}, \pm\frac{1}{1}$ span an ideal quadrilateral with one frontier geodesic in each quadrant, and one can check from the formula that the one-parameter family in each quadrant is a family of hyperbolic transformations with invariant geodesic in $\mathbb{D}$ given by the corresponding frontier geodesic of this quadrilateral as illustrated in Figure 16. In particular, $\Lambda(s)$ fixes each point $\frac{0}{1}, \frac{1}{0}, \pm\frac{1}{1}$ and hence is a homeomorphism of the circle for each $s$.

These four one-parameter families of isometries are "tuned" to guarantee that each $\Lambda(s)$ is furthermore once-continuously differentiable on $S^1_\infty$. Put another way, on the rays in $L^+$ which project to $\frac{0}{1}, \frac{1}{0}$, the function $\Lambda(s)$ acts by multiplication by $s$, and on the rays in $L^+$ which project to $\pm\frac{1}{1}$, the function $\Lambda(s)$ acts by multiplication by $s^{-1}$. We may thus think of $\Lambda(s)$ as acting on $L^+$ itself in the natural way by a piecewise $SO^+(2,1)$ homeomorphism giving a continuous one-parameter family of homeomorphisms of $L^+$. As such, each $\Lambda(s)$ acts on the lambda length of pair $h(u), h(v)$ of horocycles by $\lambda(h(u), h(v)) \mapsto \lambda(h(\Lambda(s)u), h(\Lambda(s)v))$, for $u, v \in L^+$, and hence $\Lambda(s)$ acts on $\widetilde{\mathcal{T}ess}'$.

Summarizing, we have

**Lemma 7.4.** *$\Lambda(s)$ is a multiplicative subgroup of $Homeo_+(S^1)$, where each $\Lambda(s)$ is once-continuously differentiable on $S^1$ with the four fixed points $\frac{0}{1}, \frac{1}{0}, \pm\frac{1}{1}$. If $\tau'$ is any tesselation with doe connecting $\frac{0}{1}$ to $\frac{1}{0}$ with complementary triangles spanned by $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ and $\frac{0}{1}, \frac{1}{0}, -\frac{1}{1}$, then $\Lambda(s)$ leaves invariant every lambda length coordinate of the class of $\tau'$ except that it scales the lambda length of the doe by the factor $s$.*

*Proof.* In light of the remarks above, only the last sentence requires comment. Each edge $e \in \tau'$ other than the doe has both its endpoints in some common quadrant in $\mathbb{R}^2$, and therefore Möbius invariance of lambda lengths shows that its lambda length is invariant under $\Lambda(s)$ for each $s$. On the other hand, the doe has endpoints $\frac{0}{1}$ and $\frac{1}{0}$, the points $u_1, u_2 \in L^+$ lying over these points map by $\Lambda(s)(u_j) = su_j$, for $j = 1, 2$, and hence

$$\begin{aligned}\lambda(h(u_1), h(u_2)) \mapsto &\lambda(h(su_1), h(su_2))\\ =&\sqrt{-< su_1, su_2 >}\\ =&s\sqrt{-< u_1, u_2 >}\\ =&s\lambda(h(u_1), h(u_2)).\end{aligned}$$

□

The same list of myriad questions and problems are relevant in the current decorated case as in the undecorated case. As was promised before, we finally give the unique such partial success, as follows.

An orientation-preserving mapping $f : \mathbb{D} \to \mathbb{D}$ is said to be *quasiconformal* if

$$sup_{z\in\mathbb{D}} \frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|}$$

is finite. In fact, any smooth $f$ maps infinitesimal circles to infinitesimal ellipses, and the eccentricity of such an ellipse is the expression whose supremum we take here; in particular, $f$ is conformal if and only if it maps infinitesimal circles to circles, and the expression is identically equal to one. Every quasiconformal mapping of $\mathbb{D}$ extends to an orientation-preserving homeomorphism of $S^1_\infty$, and such boundary values of a quasiconformal mapping are called *quasisymmetric* homeomorphisms of the circle. Conjugating by the Cayley transform, a more intrinsic characterization of a quasisymmetric mapping $f : \mathbb{R} \to \mathbb{R}$ is that

$$\exists M \forall x \forall t \quad M^{-1} \ < \ \lim \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \ < \ M.$$

We refer the reader to [1, 2, 22] for a detailed discussion of quasiconformal and quasisymmetric mappings including the facts just mentioned. In practice here, we shall recognize quasisymmetric mappings of $S^1_\infty$ as boundary values of quasiconformal mappings of $\mathbb{D}$.

**Example 7.5.** The characteristic mapping of the dyadic tesselation $\tau'_d$ with doe is especially interesting and is called the *Minkowski ? function.* One can see directly from the definition that it is *not* quasisymmetric, and it can be shown to be differentiable only at $\bar{\mathbb{Q}}$ with vanishing derivatives there. By definition, it conjugates the group of piecewise $PSL_2(\mathbb{Z})$ homeomorphisms of the circle to the *Thompson group $T$* of the circle. We shall have more to say about this example later (cf. Lecture 11).

We say that a function $\lambda : \tau_* \to \mathbb{R}_{>0}$ is *pinched* if

$$\exists K \ \forall e \in \tau_* \quad K^{-1} < \lambda(e) < K.$$

**Theorem 7.6.** [joint with Dennis Sullivan in [56]] *Suppose that the function $\lambda : \tau_* \to \mathbb{R}_{>0}$ is pinched. Then there is a decorated tesselation realizing $\lambda$ as its lambda length coordinates, and the corresponding homeomorphism of the circle is quasisymmetric.*

*Proof.* For the first part, since the lambda lengths are pinched, so too are the corresponding h-lengths (defined in analogy as the lengths along horocycles between geodesics) uniformly bounded above and below. The mapping from lambda lengths to cross ratios is again described by Lemma 4.9b, so the corresponding cross ratios are likewise uniformly bounded above and below. The proof of the first part is *verbatim* the same as the proof in Theorem 5.2 that the mapping from the universal cover of the surface to $\mathbb{D}$ is a homeomorphism. Indeed, one uses the lambda lengths to recursively define a mapping $\mathbb{D} \to \mathbb{D}$ pointwise fixing the triangle spanned by $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ which is a continuous injection by construction, and the facts just mentioned are used in the same manner as before to prove that this mapping is in fact a homeomorphism $\phi : \mathbb{D} \to \mathbb{D}$. This homeomorphism thus extends to an order-preserving mapping from $\tau_*^0$ to another countable dense subset of $S^1_\infty$ and therefore interpolates a normalized homemorphism $f : S^1_\infty \to S^1_\infty$ of the circle corresponding to a normalized tesselation $\tau'$ of $\mathbb{D}$. Furthermore $\tau'$ comes equipped with a decoration $\tilde{\tau}'$ determined by the lambda lengths as before.

To see that $f$ is quasisymmetric, we shall show that $f$ is given by the boundary values of a quasiconformal homeomorphism $\Phi : \mathbb{D} \to \mathbb{D}$, and we must construct $\Phi$ differently from $\phi$ to see that it is quasiconformal.

To this end, since the lambda lengths are pinched, we may scale them all by some overall factor to guarantee that they are all greater than two. It suffices to prove the result for the scaled lambda lengths, for then the original lambda lengths describe a different decoration on the same underlying tesselation, so the two corresponding homeomorphisms of the circle coincide.

Consider the Farey tesselation $\tau_*$ decorated so that each lambda length is equal to 2. The complementary regions to the union of all the horocycles in this decoration with $\cup\tau_*$ are of one of two types: either "hexagons" whose alternating sides are geodesic segments of length $2\log 2$ (by Lemma 4.1) and horocyclic segments of length $1/2$ (by Lemma 4.4), or "strips" bounded by a pair of asymptotic geodesic rays and a horocyclic segment of length $1/2$. Let $\mathcal{H}_*$ denote the union of all the hexagons and $\mathcal{S}_*$ denote the union of all the strips.

Likewise, the horocycles in the decoration of $\tilde{\tau}'$ are disjoint because of our scaling. The complementary regions to the union of these horocycles with $\cup\tau$ are again of two types: either hexagons whose alternating sides are geodesic segments of length between $2\log 2$ and $2\log 2 + 4\log K$ and horocyclic segments of length between $(2K^4)^{-1}$ and $2K^4$, or strips bounded by a pair of asymptotic geodesic rays and a horocyclic segment of length between these same latter bounds. Let $\mathcal{H}$ denote the union of all these hexagons for $\tau$ and $\mathcal{S}$ denote the union of all these strips.

There is a natural one-to-one correspondence between the hexagons in $\mathcal{H}_*$ and the hexagons in $\mathcal{H}$, and because of the bounds on lengths of geodesic and horocyclc sides, there are quasiconformal homeomorphisms uniformly near the identity on the boundaries between corresponding hexagons which combine to give a quasiconformal mapping $\Phi : \mathcal{H}_* \to \mathcal{H}$.

In order to extend to the strips, consider a strip whose horocylic segment has length $h$. Such a strip is conformal to the region in upper half-space $\mathcal{U}$ described by $\{z = x + iy : 0 \leq x \leq 1 \text{ and } h^{-1} \leq y\}$. There is thus a quasiconformal homeomorphism from the collection of strips in $\mathcal{S}$ lying inside a common horoball for $\tilde{\tau}'$ to a consecutive collection of regions $\{z = x + iy : n \leq x \leq n + 1 \text{ and } y = h_n^{-1}\}$ in $\mathcal{U}$, where $n \in \mathbb{Z}$ and the $h_n$ are uniformly bounded above and below. The corresponding region in $\mathcal{S}_*$ is conformal to $\{z = x + iy : y = 2\}$, and it is easy to extend $\Phi$ across this region preserving quasiconformality. Perform this extension for each horoball to finally construct the desired quasiconformal mapping $\Phi : \mathbb{D} \to \mathbb{D}$ with boundary values given by $f : S^1_\infty \to S^1_\infty$. $\square$

## 8. A Lie algebra for the group of circle homeomorphism

This lecture is based upon [42, 61]. We have seen in the previous lecture that each of $\mathcal{T}ess$ and $\widetilde{\mathcal{T}ess}$ have affine coordinates giving them the structure of Fréchet manifolds, and we let $tess$ and $\widetilde{tess}$ denote the respective Fréchet tangent spaces. We would like to use this to induce a reasonable Lie algebra structure corresponding to the *topological* group $Homeo_+(S^1)$ itself, and our idea is very simple: As *spaces*, we may identify $\mathcal{T}ess$ with the space $Homeo_n(S^1)$ of all normalized orientation-preserving homeomorphisms of the circle, namely, orientation-preserving homeomorphisms of the circle fixing the three points $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$. Since an element of $PSL_2(\mathbb{R})$ is determined by its values at these (or any distinct) three points, we might think that $tess \times sl_2$ is a sensible kind of tangent space to $Homeo_+(S^1)$ itself, where $sl_2$ is the Lie algebra of $SL_2(\mathbb{R})$, the double cover of $PSL_2(\mathbb{R})$.

In fact to employ lambda lengths, we shall go a bit further and define the group $\widetilde{Homeo}_n(S^1)$ of *decorated* and normalized homeomorphisms of the circle to be the set of all pairs $(\tilde{f}, f)$ with $f \in Homeo_n(S^1)$ and $\tilde{f} : \widetilde{\mathcal{T}ess} \to \widetilde{\mathcal{T}ess}$ a homeomorphism covering $f$ with the obvious group structure. In particular, there is a surjective topological group homomorphism $\widetilde{Homeo}_n(S^1) \to Homeo_n(S^1)$ gotten by projecting onto the second factor. There is furthermore an isomorphism $\widetilde{Homeo}_n(S^1) \approx \widetilde{\mathcal{T}ess}$ of $\widetilde{Homeo}_n(S^1)$-spaces gotten by assigning to

$(id, id) \in \widetilde{Homeo}_n(S^1)$ the Farey tesselation $\tau_*$ equipped with its canonical decoration, where all lambda lengths are unity. Under this isomorphism, $\widetilde{tess}$ becomes the tangent space to $\widetilde{Homeo}_n(S^1)$ at the identity. As in the previous discussion for $Homeo_n(S^1)$, we seek a natural Lie algebra structure on the product $\widetilde{tess} \times sl_2$.

Recall from Lemma 3.2 that $PSL_2(\mathbb{Z})$ acts simply transitively on the oriented edges of the Farey tesselation $\tau_*$, and let $e_A$ denote the oriented edge of $\tau_*$ which is the image $e_I A$ under the right action of $A$ on the standard doe $e_I$ connecting $\frac{0}{1}$ to $\frac{1}{0}$. Likewise, let $\frac{p}{q}A$ denote the image of the point $\frac{p}{q} \in S^1_\infty$ under $A$.

We have already in Lemma 7.4 described the basic deformation $\Lambda(s)$ of the single lambda length on the doe, and we may extend this to other edges of the Farey tesselation as follows. For each $A \in PSL_2(\mathbb{Z})$, define the corresponding

$$\Lambda_A(s) = A^{-1}\Lambda(s)A,$$

which we again may regard as a family of piecewise-$SO^+(2,1)$ homeomorphisms of $L^+$. As before, each $\Lambda_A(s)$ acts on the lambda length of pair of horocycles, and hence $\Lambda_A(s)$ acts on $\widetilde{\mathcal{T}ess}'$.

**Lemma 8.1.** *For each $A \in PSL_2(\mathbb{Z})$, $\Lambda_A(s)$ is a one-parameter multiplicative subgroup of $Homeo_+(S^1)$, where each $\Lambda_A(s)$ is once-continuously differentiable on $S^1$ with four fixed points given by $\frac{0}{1}A, \frac{1}{0}A, \pm\frac{1}{1}A$. Furthermore, for any decoration on the Farey tesselation $\tau_*$, $\Lambda_A(s)$ leaves invariant every lambda length* except *that it scales the lambda length of the unoriented edge underlying $e_A$ by the factor $s$.*

*Proof.* All the assertions for $\Lambda_A(s)$ follow from the corresponding assertions for $\Lambda(s)$ itself in Lemma 7.4. □

Thus, these one-parameter families $\Lambda_A(s)$ may be thought of as the coordinate deformations at the identity of $\widetilde{Homeo}_n(S^1)$, and we are led to consider the corresponding vector fields on $S^1$.

**Lemma 8.2.** *The vector field on $S^1$ corresponding to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in sl_2$ is given by*

$$\{(b+c)\ \cos\theta\ +\ (a-d)\ \sin\theta\ +\ (c-b)\}\frac{\partial}{\partial\theta},$$

*where $\theta$ is the usual angular coordinate on the circle.*

*Proof.* For the proof, we shall let $\begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \delta(t) \end{pmatrix}$ denote the corresponding one-parameter subgroup of $SL_2(\mathbb{R})$, so $a = \alpha'(0), \ldots, d = \delta'(0)$, where $\alpha(0) = 1 = \delta(0)$, $\beta(0) = 0 = \gamma(0)$, $\alpha'(0) + \delta'(0) = 0$, and the prime denotes the derivative with respect to $t$. The action of $\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$ corresponds to the *right* action of the Möbius group, and we conjugate its action on upper half-space by the Cayley transform to compute that $e^{i\theta} \in S^1$ maps to

$$\frac{[(\delta-\alpha)i - (\gamma+\beta)] + [(\delta+\alpha)i - (\gamma-\beta)]e^{i\theta}}{[(\delta+\alpha)i + (\gamma-\beta)] + [(\delta-\alpha)i + (\gamma+\beta)]e^{i\theta}}.$$

Take $\frac{d}{dt}|_{t=0}$ of $-i$ times the logarithm of this expression to derive the asserted formula. $\square$

We refer to a vector field as in Lemma 8.2 as a *global $sl_2$ vector field* and shall take the standard basis for $sl_2$ given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $[h,e] = 2e, [h,f] = -2f, [e,f] = h$. In this notation, the derivative of $\Lambda(s)$ is directly calculated to be

$$\vartheta = \begin{cases} \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = h + 2e, \text{ on quadrant } I; \\ \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} = -h + 2f, \text{ on quadrant } II; \\ \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} = -h - 2f, \text{ on quadrant } III; \\ \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = h - 2e, \text{ on quadrant } IV. \end{cases}$$

Writing $\vartheta = \vartheta(\theta)\frac{\partial}{\partial\theta}$, we graph this function $\vartheta(\theta)$ in Figure 17 and remark that though it "looks like" the usual sine function, it is once- but not twice-continuously differentiable.

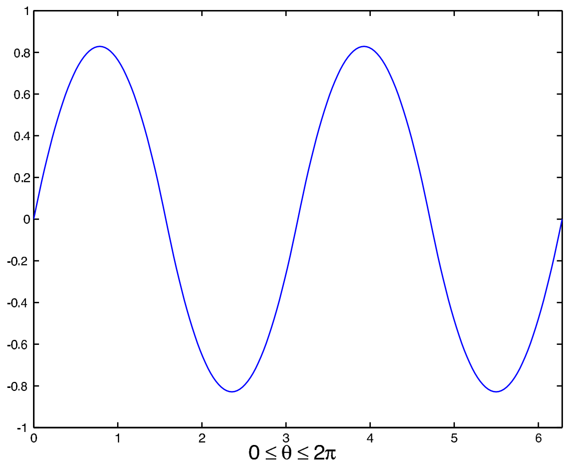



**Figure 17** Graph of The elementary vector field $\vartheta$.

The vector field $\vartheta$ lives naturally in the space $psl_2$ defined to consist of all piecewise $sl_2$ vector fields on the circle $S^1_\infty$ with finitely many pieces and with breakpoints in the piecewise structure among the rational points $\bar{\mathbb{Q}} \subset S^1$. Though $\vartheta$ is itself actually defined and continuous at its breakpoints, we do *not* require this of vector fields in $psl_2$, which are regarded as undefined at their breakpoints. (The reason for allowing these more general vector fields in $psl_2$ is that brackets of conjugates of $\vartheta$ will fail to be defined at their breakpoints as we shall see.) Given two elements of $psl_2$, there is a natural bracket defined by taking the crudest common refinement of their pieces and taking the usual bracket from $sl_2$ on each such piece. Thus, $psl_2$ is naturally a Lie algebra containing $sl_2$ as the sub-algebra of global $sl_2$ vector fields.

Having thus defined the very special element $\vartheta \in psl_2$, we proceed to define

$$\vartheta_A = A^{-1}\vartheta A, \text{ for } A \in PSL_2(\mathbb{Z}),$$

using the adjoint action on each piece. A short calculation shows that $\vartheta_S = \vartheta$, where we here and below adopt the standard notation of Lemma 3.2 for elements of the modular group $PSL_2(\mathbb{Z})$. Thus, if $e \in \tau_*$ is an *unoriented* edge, then we may associate the well-defined element $\vartheta_e = \vartheta_A = \vartheta_{SA} \in psl_2$, where $A \in PSL_2(\mathbb{Z})$ maps the unoriented edge underlying the doe of $\tau*$ to the edge $e \in \tau_*$. The vector field $\vartheta_e$ is called the *elementary vector field* associated with $e \in \tau_*$. Figure 18 illustrates the elementary vector field $\vartheta_A$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with the ideal points indicated in their Farey enumeration where the matrix near an interval indicates the corresponding element of $sl_2$ (and where we have tacitly assumed in drawing the figure that the entries of $\pm A$ are non-negative with $|c| \geq |a|$).

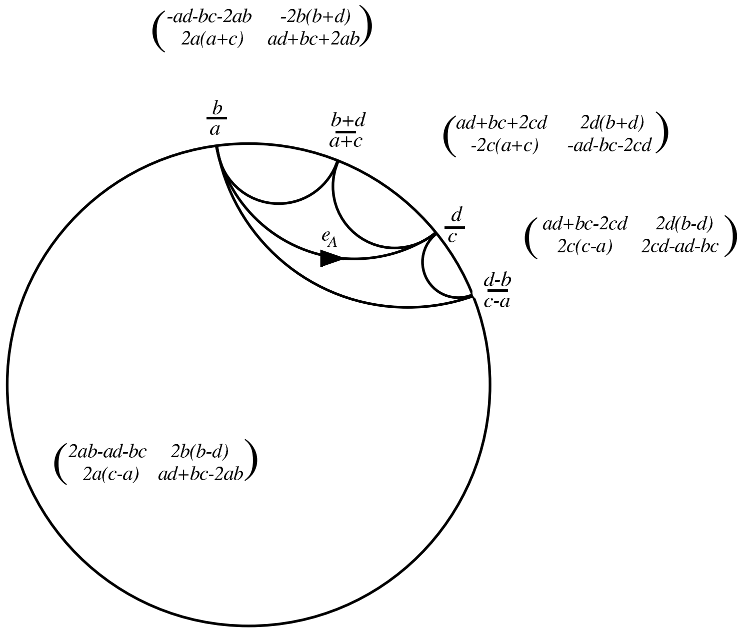



**Figure 18** The elementary vector field $\vartheta_A$.

An elementary vector field $\vartheta_A$ is defined everywhere on $S^1_\infty$, even at its breakpoints since $\vartheta$ itself has this property, so it makes sense to define the *normalization* $\bar{\vartheta}_A = \vartheta_A - x \in psl_2$ of $\vartheta_A$, where $x \in sl_2$ is chosen so that $\vartheta_A$ agrees with $x$ at the points $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$. For some examples, $\bar{\vartheta} = \vartheta$ since it already vanishes at these three points (and at $-\frac{1}{1}$ as well), and a short calculation furthermore shows that

$$\bar{\vartheta}_{U^n} = \vartheta_{U^n} + h - 2(n-1)f, \text{ for } n > 1.$$

Before giving the next definitions, let us pause to give their motivation. We wish to calculate the Lie algebra closure in $psl_2$ of $sl_2$ and the normalized elementary vector fields. Exploratory calculations of brackets of these vector fields lead to the vector fields on the circle we next introduce, which have remarkable algebraic properties. After studying these properties, we shall then prove that the Lie algebra closure we seek is actually all of $psl_2$ and defive a surprisingly simple additive basis of it.

For each oriented edge $e_A = e_I A$ of $\tau_*$ with $A \in PSL_2(\mathbb{Z})$, define the corresponding *fan vector field*

$$\phi_A = \sum_{n \geq 0} \bar{\vartheta}_{U^n A}$$

and *hyperfan vector field*

$$\psi_A = \sum_{n\geq 0} n\bar{\vartheta}_{U^n A}.$$

The initial point of the oriented edge $e_A$ is called the *pin* of the corresponding fan or hyperfan. Because of the normalization, these infinite sums converge pointwise to vector fields on the circle except perhaps at the pin, and the convergence is uniform on each compactum not containing the pin. (It is for this reason, to get such convergent sums, that we normalized the elementary vector fields.) There is, however, no reason for these vector fields to live in $psl_2$, i.e., have only finitely many pieces, and yet we have:

**Lemma 8.3.** *We have the equalities*

$$\phi_U = \begin{cases} -2e, & \text{on quadrant } I; \\ 2h - 2f, & \text{on quadrant } II; \\ 0, & \text{on quadrants } III \text{ and } IV, \end{cases}$$

$$\psi_I = \begin{cases} -2e, & \text{on quadrants } I \text{ and } II; \\ 0, & \text{on quadrants } III \text{ and } IV. \end{cases}$$

*Proof.* Letting $Ad_X(x) = X^{-1}xX$ for $x \in sl_2$ and $X \in PSL_2(\mathbb{Z})$, we first calculate that

$$Ad_U(e) = h + e - f,\ Ad_U(f) = f,\ Ad_U(h) = h - 2f.$$

The equality for the fan $\phi_U$ is proved separately in each quadrant, where one confirms that $Ad_U(h-2e) + Ad_{U^2}(-h-2f) + 2h - 2f = -2e$ for quadrant $I$, and quadrants $III$ and $IV$ are immediate. Using the expression given before for the normalizaton $\bar{\vartheta}_{U^n}$, the identity in quadrant $II$ amounts to

$$2h - 2f = [\sum_{j=1}^{n-1} Ad_{U^j}(-h+2f)] + Ad_{U^n}(h+2e)$$
$$+ Ad_{U^{n+1}}(h-2e) + [\sum_{j=1}^{n+1} h - 2(j-1)f],$$

which is proved by induction on $n \geq 2$. Likewise for the hyperfan $\psi_I$, the equality in quadrants $III$ and $IV$ is immediate, and the equality in quadrants $I$ and $II$ is tantamount to the identity

$$f - e - h = [\sum_{j=1}^{n-1} Ad_{U^j}(h - f)] - Ad_{U^n}(e),$$

which again follows by induction on $n \geq 2$. □

**Corollary 8.4.** *For any $A$ in the modular group, the fan $\phi_A$ and hyperfan $\psi_A$ lie in $psl_2$.*

*Proof.* Two fans or hyperfans are conjugate modulo an element of $sl_2$, i.e.,

$$\phi_{AB} - B^{-1}\phi_A B,\ \psi_{AB} - B^{-1}\psi_A B \in sl_2,\ \text{for } A, B \in PSL_2(\mathbb{Z}),$$

and the reason these differences do not vanish identically is that fans or hyperfans are linear combinations of *normalized* vector fields. Nevertheless, it follows from this observation and Lemma 8.3 that fans and hyperfans indeed have only finitely many pieces in their piecewise Möbius structure. □

Whereas the elementary vector fields are once continuously differentiable on the circle, fans are continuous but not differentiable, and hyperfans are not even continuous, here using that any fan is conjugate to $\phi_I$ and any hyperfan conjugate to $\psi_I$ modulo $sl_2$.

Furthermore, $\psi_I$ integrates to the square of the primitive parabolic transformation in the modular group fixing the endpoint $\frac{1}{0}$ of the doe $e_I$ and rotating counter-clockwise about this point in positive time. In general, consider the hyperfan $\psi_A$ and let $\mathcal{V}_A$ be the vector field which likewise integrates to the primitive parabolic modular transformation fixing the endpoint of $e_A$. If the triangle complementary to $\tau_*$ on the right of the doe lies to the right of the oriented edge $e_A$, then $\psi_A$ vanishes on the right of $e_A$ and agrees with $\mathcal{V}_A$ on the left; if the triangle complementary to $\tau_*$ to the right of the doe lies to the left of the oriented edge $e_A$, then $\psi_A$ vanishes on the left of $e_A$ and agrees with $-\mathcal{V}_A$ on the right. Thus, hyperfans are "piecewise parabolic" maps on the circle (in contrast to the "earthquakes introduced by Thurston [68], which are "piecewise hyperbolc").

**Theorem 8.5.** *The Lie algebra closure of $sl_2$ and the collection of all elementary vector fields $\{\bar{\vartheta}_e : e \in \tau_*\}$ is the entire Lie algebra $psl_2$.*

*Furthermore, $psl_2$ is generated as a vector space by $sl_2$ and the collection of all hyperfans.*

*Proof.* The ray from the origin in $\mathbb{R}^2$ through the point $-\frac{1}{2}$ decomposes the second quadrant $II$ into two regions denoted $II_1$, which contains $-\frac{1}{1}$ in its closure, and $II_2$, which contains $\frac{0}{1}$ in its closure. Applying the formulas in Figure 18 to the case $A = U$, we find that

$$\bar{\vartheta}_U = \begin{cases} -2e, & \text{on quadrant } I; \\ 4h - 4f + 2e, & \text{on region } II_1; \\ 4f, & \text{on region } II_2; \\ 0, & \text{on quadrants } III \text{ and } IV. \end{cases}$$

Calculating brackets, we have

$$[\bar{\vartheta}_U, e] = \begin{cases} 0, & \text{on quadrants } I, III \text{ and } IV. \\ 8e + 4h, & \text{on region } II_1; \\ -4h, & \text{on region } II_2, \end{cases}$$

$$[[\bar{\vartheta}_U, e], h] = \begin{cases} -16f, & \text{on region } II_2, \\ 0, & \text{else,} \end{cases}$$

so $[[\bar{\vartheta}_U, e], h]$ is supported on a single circular segment $C \subset S^1_\infty$. Since $sl_2$ is simple, we may likewise realize any element of $sl_2$ on $C$, and in particular, both $\psi_{TU}$ and $\psi_{STU}$, which are supported on $C$, lie in the Lie algebra generated by the elementary vector fields and $sl_2$. Since this Lie algebra is by definition closed under the adjoint action of $SL_2(\mathbb{R})$, it follows that every hyperfan lies in this algebra.

To complete the proof, we must show that $sl_2$ and hyperfans span $psl_2$ as a vector space. To this end, consider the circular segment in $S^1_\infty$ whose endpoints are given by the ideal points of $e_A$, where the entries of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ are non-negative. Let $C$ denote the circular sub-segment bounded by the ideal points of $e_{UA}$. By the previous characterization of hyperfans, $\psi_A$ and $\psi_{TA}$ take common values on the intersection of their supports, and $\psi_{UA}, \psi_{SUA}, (\psi_A - \psi_{TA})$ all have support $C$, and furthermore, these vector fields restricted to their common support $C$ are given by

$$\begin{aligned}\psi_{UA} &= -2(b+d)^2\ e\ +\ 2(a+c)^2\ f\ -\ 2(a+c)(b+d)\ h,\\ \psi_{SUA} &= 2b^2\ e\ -\ 2a^2\ f\ +\ 2ab\ h,\\ (\psi_A - \psi_{TA}) &= -2d^2\ e\ +\ 2c^2\ f\ -\ 2cd\ h.\end{aligned}$$

The matrix

$$\begin{pmatrix} -2(b+d)^2 & 2(a+c)^2 & -2(a+c)(b+d) \\ 2b^2 & -2a^2 & 2ab \\ -2d^2 & 2c^2 & -2cd \end{pmatrix}$$

is found to have determinant $8(ad-bc)^3 = 8 \neq 0$, and hence any vector field supported on $C$ lies in the span of these. A similar argument using the hyperfans $\psi_{SA}, \psi_{STA}, \psi_{SUA}$ likewise handles the circular subsegment bounded by the ideal points of $e_{TA}$, and indeed the analogous argument for a circular segment lying in the lower half-plane completes the proof. □

*Remark* 8.6. In fact, one can give a complete set of additive relations among hyperfans, namely, for any $A \in PSL_2(\mathbb{Z})$, we have

$$\psi_{U^{-1}A} - 2\psi_A + \psi_{UA} \ = \ \psi_{U^{-1}SA} - 2\psi_{SA} + \psi_{USA}.$$

Indeed from the definitions of fans and hyperfans, we find

$$\begin{aligned}\phi_A - \phi_{UA} &= \bar{\vartheta}_A = \phi_{SA} - \phi_{USA},\\ \psi_A - \psi_{UA} &= \phi_A,\ \psi_{SA} - \psi_{USA} = \phi_{SA}\end{aligned}$$

arising from the identity $\vartheta_A = \vartheta_{SA}$ discussed before, which thus yields the asserted relations among hyperfans. Furthermore, these relations are linearly independent, and they span the space of linear dependencies among hyperfans, cf. Proposition 6 of [42]. One can use this result together with further calculations to give an additive basis for $psl_2$, namely, $e, f, h$ and

$$\{\psi_A : A \in \{I\}\ \cup\ U\mathcal{P} \cup U^{-1}\mathcal{N}\ \cup ST\mathcal{P} \cup ST^{-1}\mathcal{N},$$

where $\mathcal{P}$ is the monoid of products of $T, U$ and $\mathcal{N}$ is the monoid of products of $T^{-1}, U^{-1}$, cf. Theorem 2.1 of [61]. We shall not prove these facts here since the arguments in print are somewhat involved and yet are the simplest we know. In contrast, we have included the proof of Theorem 8.5 since it is much simpler than the previously published proof.

This completes our algebraic discussion of the Lie algebra $psl_2$ that arises by considering lambda lengths as coordinate deformations of the space $\widetilde{Homeo}_n$ of decorated normalized homeomorphisms of the circle, and we finally turn our attention to the analysis of these deformations.

Let $Diff_+(S^1) \subset Homeo_+(S^1)$ denote the subgroup of infinitely differentiable orientation-preserving homeomorphisms of the circle, whose Lie algebra $diff_+$ consists of all infinitely differentiable real vector fields $f(\theta)\frac{\partial}{\partial\theta}$ of the circle, which we may Fourier expand

$$\begin{aligned} f(\theta) &\sim \sum_{n\geq 0} a_n \cos n\theta \ + \ b_n \sin n\theta \\ &= \sum_{n\in\mathbb{Z}} c_n \ e^{in\theta}, \text{ with } c_n = \bar{c}_n. \end{aligned}$$

Just as for homeomorphisms, let $Diff_n(S^1) = Diff_+(S^1) \cap Homeo_n(S^1)$ denote the subgroup of normalized diffeomorphisms fixing $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}$ with its Lie algebra $diff_n$ of all infinitely differentiable vector fields which vanish at these three points. Given any vector field $\mathcal{V} = f(\theta)\frac{\partial}{\partial\theta}$ defined on $S^1$, we define its *normalization*

$$\bar{\mathcal{V}} = \bar{f}(\theta)\frac{\partial}{\partial\theta} = \{f(\theta) + \alpha \cos\theta + \beta \sin\theta + \gamma\}\frac{\partial}{\partial\theta},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are chosen so that $\bar{f}(\theta) = 0$ for $\theta = 0, \pi, 3\pi/2$. For example with $z = e^{in\theta}$, a calculation gives the following *normalized trigonometric fields* for $n \geq 2$:

$$\overline{\cos}\ n\theta\frac{\partial}{\partial\theta} = \frac{1}{2}\left\{i(z^n + z^{-n}) + \alpha i(z + z^{-1}) + \beta(z - z^{-1}) + 2\gamma i\right\} z\frac{d}{dz},$$

with

$$\begin{aligned} &2\alpha = (-1)^n - 1,\ 2\gamma = -1 - (-1)^n, \\ &2\beta = (-1 - (-1)^n) - (-1)^{n+1}(i^n + i^{-n}), \end{aligned}$$

and

$$\overline{\sin}\ n\theta\frac{\partial}{\partial\theta} = \frac{1}{2}\left\{z^n - z^{-n} + \alpha i(z + z^{-1}) + \beta(z - z^{-1}) + 2\gamma i\right\} z\frac{d}{dz},$$

with

$$\alpha = 0 = \gamma,\ 2\beta = i(-1)^{n+1}(i^n - i^{-n}),$$

which constitute a natural basis for $diff_n$ that we shall require later. Put another way, the exponential functions $e^{in\theta}$ admit the normalization

$$\bar{e}^{in\theta} = e^{in\theta} - [b_0^n + b_1^n e^{i\theta} + b_{-1}^n e^{-i\theta}],$$

where

$$b_0^n = \begin{cases} +1, n \equiv 0(4); \\ 0, n \equiv 1(4); \\ +1, n \equiv 2(4); \\ 0, n \equiv 3(4); \end{cases} \quad b_1^n = \begin{cases} 0, n \equiv 0(4); \\ +1, n \equiv 1(4); \\ -i, n \equiv 2(4); \\ 0, n \equiv 3(4); \end{cases} \quad b_{-1}^n = \begin{cases} 0, n \equiv 0(4); \\ 0, n \equiv 1(4); \\ +i, n \equiv 2(4); \\ +1, n \equiv 3(4). \end{cases}$$

**Theorem 8.7.** *If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z})$, then $\bar\vartheta_A \sim \sum_{n\in\mathbb{Z}} c_n\, e^{in\theta}\frac{\partial}{\partial\theta}$, where for $n^2 > 1$, we have*

$$\begin{aligned} \pi i(n^3-n)\ c_n \ &= \ -[(c-a)^2+(b-d)^2]\left[\frac{(b-d)-i(a-c)}{(b-d)+i(a-c)}\right]^n \\ &\quad + 2(c^2+d^2)\left[\frac{d-ic}{d+ic}\right]^n \ + \ 2(a^2+b^2)\left[\frac{b-ia}{b+ia}\right]^n \\ &\quad - [(c+a)^2+(b+d)^2]\left[\frac{(b+d)-i(a+c)}{(b+d)+i(a+c)}\right]^n, \end{aligned}$$

*and the Fourier modes $c_0, c_{\pm 1}$ are chosen to guarantee that the expansion is normalized, i.e., $\bar\vartheta_A \sim \sum_{n^2>1} c_n\ \bar{e}^{in\theta}\frac{\partial}{\partial\theta}$.*

*Proof.* Since the normalized elementary vector field $\bar\vartheta_A$ is once continuously differentiable on the circle, we may twice integrate by parts the usual expression for Fourier coefficients to conclude that

$$c_n = \frac{1}{2\pi i}\ \frac{1}{n^3-n}\ \sum_j z_j\ e^{in\theta}|_{\partial I_j},\ \text{for } n^2>1,$$

where $I_j$, for $j = 1,2,3,4$, are the intervals in the $psl_2$ structure of $\bar\vartheta_A$ and $\bar\vartheta_A = \{x_j\ \cos\theta + y_j\ \sin\theta + z_j\}\frac{\partial}{\partial\theta}$ on $I_j$.

We may calculate these zero-modes $z_j$ explicitly from Figure 18 for $A \neq I, S$, where there are actually four separate cases depending upon in which quadrant lies $e_A$. In each case, the explicit not-entirely-painless calculation yields the asserted expression using that the Farey point $\frac{p}{q} \in S^1_\infty$ corresponds to the complex number $\frac{p+iq}{p-iq}$. Notice that for $n = 0, \pm 1$ both sides of the asserted expression vanish identically. In particular for the elementary vector field $\vartheta_I$, the analogous direct calculation gives $\vartheta_I \sim \frac{8}{\pi i}\ \sum_{n\equiv 2(4)}\ \frac{1}{n^3-n}\ e^{in\theta}\ \frac{\partial}{\partial\theta}$, which likewise agrees with the asserted formula. □

This satisfactorily describes the harmonic analysis of *homeo*$_+$, and we finally wish to understand the reverse inclusion. To this end and to employ lambda lengths, define $\widetilde{Diff}_n(S^1)$ to be the collection of all pairs $(\tilde{f}, f)$ , where $f \in Diff_n(S^1)$ and $\tilde{f}$ is a homeomorphism of $\widetilde{\mathcal{T}ess}$ covering $f$; there is natural group structure on $\widetilde{Diff}_n(S^1)$ as before and a natural surjective group homomorphism $\widetilde{Diff}_n(S^1) \to Diff_n(S^1)$.

We define a section

$$\sigma : Diff_n(S^1) \to \widetilde{Diff}_n(S^1)$$

as follows. If $f \in Diff_n(S^1)$, we may conjugate by the Cayley transform $C : \mathcal{U} \to \mathbb{D}$ to produce a diffeomorphism $f^C = C^{-1} \circ f \circ C : \mathbb{R} \to \mathbb{R}$, where $f^C$ fixes $0, 1 \in \mathbb{R}$. A horocycle in $\mathcal{U}$ tangent to $x \in \mathbb{R}$ is determined by its Euclidean diameter $d$, and we define the Euclidean diameter of the horocycle for $\sigma(f)$ at $f^C(x)$ to be $|\frac{df^C}{dx}(x)|d$, where this definition is motivated by Lemma 4.2. Likewise, define the the evolution under $f^C$ of a horocycle in $\mathcal{U}$ centered at infinity so that its Euclidean height scales by the reciprocal of the derivative of $f^C$ at infinity. This defines a natural action of $Diff_n(S^1)$ on decorations and hence defines the desired section $\sigma$. The chain rule then shows that $\sigma$ is furthermore a group homomorphism.

This section allows us to regard

$$Diff_n(S^1) \approx \sigma(Diff_n(S^1)) \subset \widetilde{Diff}_n(S^1) \subset \widetilde{Homeo}_n(S^1)$$

so as to calculate with lambda lengths, that is, we shall finally calculate $diff_n \subset \widetilde{tess}$ using normalized trigonometric vector fields as follows.

Let $w(z,t)$ denote the one-parameter family of diffeomorphisms of $S^1$ that arise by integrating the normalized trigonometric fields $\overline{\cos} n\theta \frac{\partial}{\partial\theta}$ and $\overline{\sin} n\theta \frac{\partial}{\partial\theta}$ discussed before. In particular, we have $\frac{\partial w}{\partial z}|_{t=0} = 1$, an identity we shall serially apply in the sequel. To calculate $\sigma$, define

$$z = C(s) = \frac{s-i}{s+i}, \text{ for } s \in \mathbb{R},$$
$$s = C^{-1}(z) = i\frac{1+z}{1-z}, \text{ for } z \in S^1_\infty,$$

so that

$$\begin{aligned} W(s,t) &= C^{-1} \circ w(C(s), t) \\ &= i\,\frac{1 + w(\frac{s-i}{s+i}, t)}{1 - w(\frac{s-i}{s+i}, t)} : \mathbb{R} \to \mathbb{R} \end{aligned}$$

and in particular $\frac{\partial W}{\partial s}\Big|_{t=0} = 1$.

By Lemma 4.2, the lambda lengths are

$$\lambda(x,y) = \frac{|y-x|}{\sqrt{d_x\, d_y}}$$

if the horocycles at $x, y \in \mathbb{R}$ have respective diameters $d_x, d_y$. By definition of the section $\sigma$, the lambda length $\lambda(x, y)$ evolves under $W(\cdot, t)$ to

$$\lambda_W(x,y,t) = \frac{\mid W(y,t) - W(x,t) \mid}{\sqrt{\left(\left|\frac{\partial W}{\partial s}\right|_x d_x\right) \left(\left|\frac{\partial W}{\partial s}\right|_y d_y\right)}},$$

and so we seek

$$\begin{aligned}
\frac{\partial}{\partial t}\bigg|_{t=0} \left\{ \log \lambda_W(x,y,t) \right\} &= \frac{\frac{\partial}{\partial t}\big|_{t=0} \left\{ \lambda_W(x,y,t) \right\}}{\lambda_W(x,y,0)} \\
&= \frac{\frac{\partial}{\partial t}\big|_{t=0} \left\{ \lambda_W(x,y,t) \right\}}{\lambda(x,y)} \\
&= \frac{1}{|y-x|} \frac{\partial}{\partial t}\bigg|_{t=0} \left\{ \frac{|W(y,t) - W(x,t)|}{\sqrt{\left|\frac{\partial W}{\partial s}\right|_x} \sqrt{\left|\frac{\partial W}{\partial s}\right|_y}} \right\}.
\end{aligned}$$

Direct calculation using the identity $\frac{\partial W}{\partial s}\big|_{t=0} = 1$ gives

$$\frac{\partial}{\partial t}\bigg|_{t=0} \left\{ \log \lambda_W(x,y,t) \right\} =$$
$$\left[ \frac{1}{|y-x|} \left| \frac{\partial W}{\partial t}\Big|_y - \frac{\partial W}{\partial t}\Big|_x \right| - \frac{1}{2} \left\{ \frac{\partial^2 W}{\partial s \partial t}\Big|_x + \frac{\partial^2 W}{\partial s \partial t}\Big|_y \right\} \right]_{t=0}.$$

We may substitute into the previous expression the known derivatives

$$\frac{\partial W}{\partial t}(s,t) = \frac{2i}{[1 - w(\frac{s-i}{s+i}, t)]^2} \frac{\partial w}{\partial t}\left(\frac{s-i}{s+i}, t\right),$$

so at $t = 0$, we find

$$\frac{\partial W}{\partial t}(s,0) = \frac{2i}{[1 - \frac{s-i}{s+i}]^2} \frac{dz}{dt}\bigg|_{z=\frac{s-i}{s+i}} = -\frac{i}{2} (s+i)^2 \frac{dz}{dt}\bigg|_{z=\frac{s-i}{s+i}},$$

where $\frac{dz}{dt}$ is the coefficient of $\frac{d}{dz}$ in the normalized trigonometric fields described before.

In case one of the points, say the point $y$, lies at infinity, a parallel calculation to that above again using Lemma 4.2 leads to

$$\frac{\partial}{\partial t}\bigg|_{t=0} \left\{ \log \lambda_W(x,\infty,t) \right\} = -\frac{1}{2} \left\{ \frac{\partial^2 W}{\partial s \partial t}\Big|_x + \frac{\partial^2 W}{\partial s \partial t}\Big|_\infty \right\} \bigg|_{t=0},$$

which is the limit of the earlier formula for fixed $x \in \mathbb{R}$ as $y \to \infty$.

**Theorem 8.8.** *For each $n \in \mathbb{Z}$, we have the expansion*

$$e^{in\theta}\frac{\partial}{\partial\theta} = b_0^n + b_{+1}^n e^{i\theta} + b_{-1}^n e^{-i\theta} + \frac{i}{4}\sum_{e\in\tau_*}\left\{n(\xi^n+\eta^n) + \frac{\eta+\xi}{\eta-\xi}\,(\xi^n-\eta^n)\right\}\bar{\vartheta}_e(\theta),$$

*where $e \in \tau_*$ has ideal points $\xi, \eta \in S^1$.*

*Proof.* A series of calculations in the various cases for the residue of $n$ modulo four lead from the previous expression to the formulas

$$\overline{\sin}\, n\theta\,\frac{\partial}{\partial\theta} \sim \frac{1}{8}\sum_{e\in\tau_*}\frac{(\xi^n\eta^n+1)}{\xi^n\eta^n}\left\{n(\xi^n+\eta^n) + \frac{\eta+\xi}{\eta-\xi}\,(\xi^n-\eta^n)\right\}\bar{\vartheta}_e,$$

$$\overline{\cos}\, n\theta\,\frac{\partial}{\partial\theta} \sim \frac{i}{8}\sum_{e\in\tau_*}\frac{(\xi^n\eta^n-1)}{\xi^n\eta^n}\left\{n(\xi^n+\eta^n) + \frac{\eta+\xi}{\eta-\xi}\,(\xi^n-\eta^n)\right\}\bar{\vartheta}_e,$$

which turn out to be independent of the case and combine to give the asserted formulas for exponentials. $\square$

Let us identify the vector field $\bar{\vartheta}_e = \bar{\vartheta}_e(\theta)\frac{\partial}{\partial\theta}$ with the corresponding function $\bar{\vartheta}_e(\theta)$ of the same name. Motivated by harmonic analysis, it is natural to try to expand a general function on the circle using these functions. Basic questions include the convergence and uniqueness of such expansions. Because of the supports of the functions $\bar{\vartheta}$, we see that any linear combination $\sum_{e\in\tau_*} c_e\bar{\vartheta}_e(\theta)$ necessarily takes well-defined values at the points $\bar{\mathbb{Q}} \subset S^1$, so any expansion converges at least in this sense. On the other hand, for any such expansion and any $\mathbb{R}$-valued function $F$ defined on $\bar{\mathbb{Q}}$, we likewise see that the alternate expansion $\sum_{e\in\tau_*}(c_e + F(\xi) + F(\eta))\bar{\vartheta}_e(\theta)$ takes the same values on $\bar{\mathbb{Q}}$, where $\xi, \eta \in S^1$ again denote the ideal points of $e \in \tau_*$. The expansion given in Theorem 8.8 is thus *not* the simplest possible; for instance, we could simply omit the term $n(\xi^n + \eta^n)$ without affecting the values on $\bar{\mathbb{Q}}$. The basic estimates from [61], expressed in terms of the invariant $\lambda_A = ac + bd$ of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, are that there are constants *const* so that:

if $g_n^e$ is the coefficient of $\bar{\vartheta}_e$ in the expansion in Theorem 8.8 of $e^{in\theta}$, then we have $|g_n^e| < const\; n^2\; |\lambda_A|^{-1}$ and $|g_n^{e_I}| < |n|/2$;

$\sum |\lambda_A|^{-2} < \infty$, where the sum is over all $A \in PSL_2(\mathbb{R}) - \{I, S\}$;

the $L^2$-norm of $\bar{\vartheta}_A$ is less than *const* $|\lambda_A|^{-\frac{5}{2}}$;

for every $\theta$, we have $|\bar{\vartheta}_A(\theta)| < const\ |\lambda_A|^{-1}$.

Various results about the function theory of expansions in $\{\bar{\vartheta}_e\}$ can be derived from these facts.

A fundamental point is that because of the supports of the functions $\bar{\vartheta}_e$, the coefficients $c_e$ in an expansion $f(\theta) \sim \sum_{e\in\tau_*} c_e\ \bar{\vartheta}_e(\theta)$ can be recursively calculated from the values of the function $f(\theta)$ at the points $\bar{\mathbb{Q}}$ in their Farey enumeration. Furthermore, because the harmonic expansion of the functions $\bar{\vartheta}_e(\theta)$ is known from Theorem 8.7, this provides a viable alternative method of harmonic analysis based on this method of "Farey sampling". For better or worse, we have actually successfully patented these methods of signal processing whose practical application depends upon data sampled at the Farey points.

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