RIGID ACTIONS OF MAPPING CLASS GROUPS

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Contents

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INTRODUCTION

These notes grew out of a master class I gave in April 2007 at the Center for the Topology and Quantization of Moduli spaces of the University of Aarhus.[1] The subject of the master class was the actions of (extended) mapping class groups of surfaces on spaces of different sorts. Thes space encode various geometric and topological objects on the surfaces, like homotopy classes of curves, of foliations, of conformal structures and of metrics of constant curvature. The actions of the mapping class groups on these spaces are all induced from the actions of homeomorphisms of the surfaces on corresponding objects. Moreover, the spaces on which the mapping class groups act are equipped with various structures, namely, they are groups, simplicial complexes, analytic varieties, Finsler or Riemannian manifolds, and the mapping class groups are embedded accordingly into groups of algebraic isomorphisms, of simplicial automorphisms, of holomorphic (and anti-holomorphic) automorphisms, and of isometries of the various metrics. The leitmotiv in this study is that for most of these actions, the natural homomorphism from the mapping class group of the surface in the automorphism group of the given structure is an isomorphism, if one excludes some special surfaces with small genera and small number of boundary components. (In general the exceptional surfaces are the spheres with at most four holes, the tori with at most two holes, and the closed surface of genus two).
Here is a detailed list of the actions which I describe in these notes:

(1) Algebraic actions: These include the action of the mapping class group by linear automorphisms on the first homology group of the surface, the action of the extended mapping class group by outer automorphisms of the fundamental group of the surface, the action of the extended mapping class group on itself by inner automorphisms, and the action of the extended mapping class group on the Torelli group by conjugation.
(2) The action of the extended mapping class group by biholomorphic and anti-biholomorphic automorphisms on the Teichmüller space of

[1]I would like to thank Jørgen Ellegaard Andersen who invited me to give this master class.

the surface, equipped with the complex analytic structure of that space that was investigated by Ahlfors and Bers.
(3) Actions of the extended mapping class group by isometries on Teichmüller space with respect to various metrics (the Teichmüller metric, the Weil-Petersson metric and Thurston's asymmetric metric.)
(4) Actions of the extended mapping class group by simplicial automorphisms on various abstract simplicial complexes associated to the surface (the curve complex, the pants decomposition complex, the complex of nonseparating curves, the complex of domains, etc.).
(5) The actions of the extended mapping class group on the Hatcher-Thurston and on the pants 2-dimensional cell complexes.
(6) The actions of the extended mapping class group by piecewise linear automorphisms of measured foliations space, equipped with the train track piecewise linear structure introduced by Thurston.
(7) The action of the extended mapping class group by homeomorphisms on the space of equivalence classes of measured foliations equipped with the set of intersection functions, and preserving this set.
(8) The action of the extended mapping class group by homeomorphisms of the non-Hausdorff space of unmeasured foliations.

There are other actions of the (extended) mapping class group which are certainly worth studying and which I do not mention in these notes, except sometimes for stating an open problem. Examples of such actions include the action by isometries on metrics other than the Teichmüller and the Weil-Petersson (e.g. McMullen's metric, the Kähler-Einstein metric, etc.), the algebraic action by conjugation on other normal subgroups than the Torelli group, the action on the character variety of representations of the fundamental group of the surface in $\mathrm{SL}(2,\mathbb{C})$ and in other Lie groups that preserve the trace functions, the action by symplectomorphisms of Teichmüller space with respect to the Weil-Petersson symplectic structure, the action by homeomorphisms of Teichmüller space that preserve the set of hyperbolic length functions, the actions on boundaries other than Thurston's boundary (for instance, the action on the Gromov boundary of the complex of curves), and the actions on quantum Teichmüller and measured foliation spaces. One could also study actions analogous to those that are presented here for surfaces that are not of finit type, see e.g. the results obtained by Markovic for the isometries of the Teichmüller metrics of such surfaces, cf. [42] and the survey in [18]. There are also actions analogous to those of the mapping class group on the soleniod, see e.g. the work by Bonnot, Penner & Šarić [6] and the survery by Šarić [65], and the list could go on and on.
Each of the actions that I mention in these notes has a particular geometric flavour, and as I already mentioned, a constant feature of these actions is that, in most cases, the representation of the (extended) mapping class group in the automorphism group of the structure is *faithful*, that is, the mapping class group injects into the automorphism group, except, as we already said,

for a small finite set of surfaces (notably, the case of the closed surface of genus two or the torus with one or two holes, or the sphere with four holes, where the hyperelliptic involutions are in the kernel of the representation), cf. Figure 1. More interestingly, there are quite a number of cases where the representation is also surjective, that is, where every automorphism of the structure is induced by an element of the extended mapping class group.

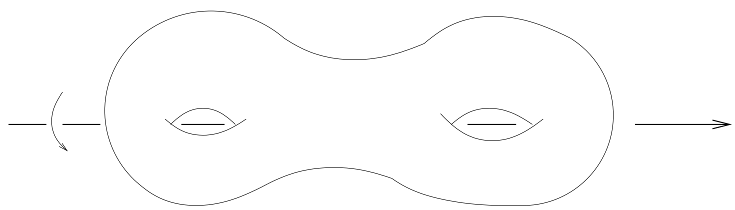

FIGURE 1. The hyperelliptic involution of the closed surface of genus two can be seen as the 180-degree rotation about the horizontal axis.

We call an automorphism of a given structure *geometric* if it is induced by an element of the extended mapping class group. We shall say that the action is *rigid* if the associated automorphism group coincides with the (extended) mapping class group. This is the feature which is highlighted in these notes. As was already mentioned, in general, in order for the action of the mapping class group to be rigid, one has to exclude some special surfaces, and for the majority of the actions that are considered here, the excluded surfaces are surfaces with low genus and with a small number of boundary components. But there are also actions for which one has to exclude a large family of surfaces. For instance, the action of the extended mapping class group as outer automorphisms of the fundamental group is rigid if and only if the surface has no boundary. Another example that we shall study and which has a different character is the case of the action of the extended mapping class group by simplicial automorphisms on the complex of domains. In that case, the action is rigid if and only if the surface has at most one boundary component.

In any case, it is a natural and interesting question to find exactly for which surfaces the automorphism group of the structure under consideration is naturally identified with the extended mapping class group. Moreover, for the excluded surfaces have low genus and a small number of boundary components, it is generally possible and instructive to obtain a complete geometric picture of the structure, to describe its full automorphism group and to see exactly to what extent this group fails to be equal to the natural image of the extended mapping class group in that group.

In these notes, I describe in detail the following actions:

- the simplicial action on the complex domain, which is recent joint work with John McCarthy;

- the piecewise linear action on the train track piecewise-linear structure on the space of equivalence classes of measured foliations;
- the action by homeomorphisms on the non-Hausdorff space of unmeasured foliations.

Concerning the other actions, I state the corresponding rigidity and/or non-rigidity results, after a review of the definitions and of some background material that is needed to understand them, together with a discusssion of the special cases, but with a minimum amount of technicalities.
I start with a few definitions which will be useful throughout these notes.
We shall always denote by $S = S_{g,n}$ a connected oriented compact surface of genus $g \geq 0$ with $n \geq 0$ boundary components. In the case where S is closed (that is, if $b = 0$), we shall also write $S = S_g$.
Let $\mathrm{Homeo}(S)$ be the set of all homeomorphisms of S, equipped with the group structure defined by composition of maps.
We need to consider a topology on the group $\mathrm{Homeo}(S)$, because we want to talk about homotopies in that space. We equip $\mathrm{Homeo}(S)$ with the topology of uniform convergence. (We shall equip $\mathrm{Homeo}(S)$ with the compact open topology in later sections where we consider surfaces with a finite number of points deleted instead of compact surfaces with boundary.)
If f and g are two elements of $\mathrm{Homeo}(S)$, then, we shall call an *isotopy* from f to g a continuous map $H : [0,1] \to \mathrm{Homeo}(S)$ satisfying $H_0 = f$ and $H_1 = g$.
Let $\mathrm{Homeo}_0(S)$ be the group of homeomorphisms of S that are isotopic to the identity. In other words, $\mathrm{Homeo}_0(S)$ is the connected component of the identity in the group $\mathrm{Homeo}(S)$.
Finally, let $\mathrm{Homeo}^+(S)$ be the space of homeomorphisms of S that preserve the orientation of S, and $\mathrm{Homeo}_0^+(S)$ be the intersection $\mathrm{Homeo}_0(S) \cap \mathrm{Homeo}^+(S)$.
The subgroups $\mathrm{Homeo}_0(S)$ and $\mathrm{Homeo}_0^+(S)$ are normal subgroups of $\mathrm{Homeo}(S)$ and $\mathrm{Homeo}^+(S)$ respectively.
The *extended mapping class group of* S, denoted by Γ^*, $\Gamma^*_{g,n}$ or $\Gamma^*(S)$, is the quotient group

$$\Gamma^*(S) = \mathrm{Homeo}(S)/\mathrm{Homeo}_0(S).$$

The *mapping class group of* S, denoted by Γ, $\Gamma_{g,n}$ or, $\Gamma(S)$, is the quotient group

$$\Gamma(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0^+(S).$$

In other words, the extended mapping class group of S is the set of isotopy classes of homeomorphisms of S, and the mapping class group of S is the set of isotopy classes of orientation-preserving homeomorphisms of S. The two sets are equipped with the group operation that is induced by composition of maps.
Elements of the (extended) mapping class group are called *(extended) mapping classes*.
Note that $\Gamma(S)$ is a normal subgroup of index 2 in $\Gamma^*(S)$.

Any homeomorphism of S preserves the boundary components of S. Note that we do not ask that it preserves each boundary component. I mention here this fact because in the literature on mapping class groups, some authors deal with homeomorphisms that preserve each boundary component, and even, in some contexts, the authors ask that the homeomorphisms induce the indentity map on each boundary component. For instance, this is the case when one is interested in homomorphisms between mapping class groups of sequences of nested surfaces. Groups of mapping classes that preserve setwise each boundary components are normal subgroups of $\Gamma(S)$ and $\Gamma^*(S)$. We note in this respect that in his paper *The group of mapping classes* [8], in which Max Dehn describes a finite set of generators for mapping class groups consisting of (what we call now) Dehn twists, Dehn discusses various possibilities for the homeomorphisms and the homotopies defining the mapping class groups in the case of surfaces with boundary: possibly exchaging the boundary components, or fixing setwise each boundary component, or fixing pointwise the points on the boundary and so on (cf. [8] p. 256 ff.).

We also note that in the definition of the (extended) mapping class group, one can use diffeomorphisms instead of homeomorphisms. This requires taking an arbitrary differentiable structure on the surface, and noting that the group obtained does not depend on the choice of the differentiable structure. Therefore this group is canonically isomorphic to the one defined by using homeomorphisms. Using diffeomorphisms instead of homeomorphisms can seem somehow artificial, because the differentiable structure is generally not useful in the theory (except for instance when one deals with conformal structure on S, but in this case the differentiable structure comes as a by-product of the conformal structure). But a differentiable structure can sometimes have advantages; for instance, it is easier to talk about an orientation-preserving diffeomorphisms rather than an orientation-preserving homeomorphisms. (Note that one can define a homeomorphism to be "orientation-preserving" if it is isotopic to a diffeomorphism that is orientation-preserving.) The mapping class group is sometimes called the *diffeotopy group*. (This is also the french terminology.)

There are some surfaces of small genus and with a small number of boundary components, of which the (extended) mapping class groups have easy decriptions. Let us mention the following cases:

1) The mapping class group of the sphere $S^2 = S_{0,0}$ is trivial. This is a consequence of a theorem of Smale saying that the group of orientation-preserving diffeomorphisms $\mathrm{Diffeo}(S^2)$ of the two-sphere is homotopy-equivalent to its subgroup $O(3)$ of rotations, and therefore, $\mathrm{Diffeo}(S^2)$ is connected. The extended mapping class group of the sphere is $\mathbb{Z}_2$, generated by the isotopy class of any orientation-reversing homeomorphism.

2) We have $\Gamma(S_{0,1}) = \{\mathrm{Id}\}$, $\Gamma^*(S_{0,1}) \simeq \mathbb{Z}_2$, $\Gamma(S_{0,2}) = \mathbb{Z}_2$ and $\Gamma^*(S_{0,2}) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Note that in the case of the disk $S_{0,1}$, there exists a homotopy of maps of the disk which reverses the orientation of the surface, but this is

not an isotopy in the sense we are using this word here. (Remember that an isotopy is a continuous path of homeomorphisms, and each homeomorphism preserves the boundary of the surface). Likewise, in the case of the annulus, there exists a homotopy (but not an isotopy) between the identity map and a homeomorphism which interchanges the two boundary components and reverses the orientation, but again, this is not an isotopy. In any case, the disk and the annulus are the only two surfaces where such phenomena can happen.
3) If $S = S_{0,3}$ (a pair of pants), then $\Gamma(S)$ is the permutation group on three elements (the three boundary components), and $\Gamma^*(S)$ is a degree-two extension of $\Gamma(S)$ by any orientation-reversing homeomorphims of S.
4) If $S = S_1$ (the torus) or $S = S_{1,1}$ (the torus with one hole), the mapping class group $\Gamma(S)$ is infinite, and it is generated by the two Dehn twists represented respectively in the two pictures on the right in Figure 2. We have isomorphisms $\Gamma(S_{1,0}) \simeq \Gamma(S_{1,1}) \simeq \mathrm{SL}(2,\mathbb{Z})$ and $\Gamma^*(S_{1,0}) \simeq \Gamma^*(S_{1,1}) \simeq \mathrm{GL}(2,\mathbb{Z})$. Again, the extended mapping class group is obtained by taking an extension of the mapping class group by the isotopy class of any orientation-preserving homeomorphism.
5) In the case where $S = S_{0,4}$, then $\Gamma(S)$ is an infinite group generated by the Dehn twists around the two curves represented in the picture to the left in Figure 2 and $\Gamma^*(S)$ is an extension of $\Gamma(S)$ by any orientation-reversing mapping class. These groups are finite extensions of the mapping class group and (respectively) the extended mapping class group of the torus.

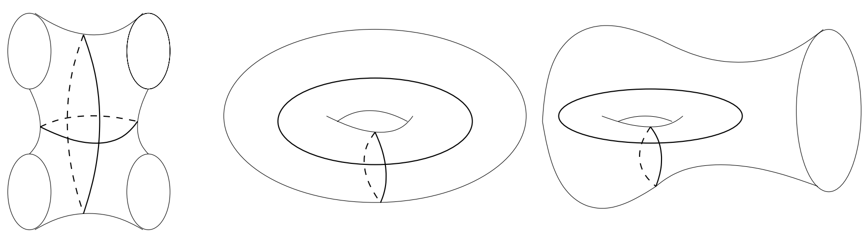

FIGURE 2. For each of the surfaces represented, the mapping class group is generated by Dehn twists around the two curves drawn on them.

We finally note that in the case of surfaces of genus 0, mapping class groups have several special algebraic descriptions as they essentially coincide with braid groups.

Part 1. **ALGEBRAIC ACTIONS**

1. Automorphisms of homology

In this section, S is a closed surface. We study the classical and well understood action of the mapping class group of S on the first singular homology group of S.(For reasons that are apparent in the analysis of the action, it is

natural to deal here with the mapping class group and not with the extended mapping class group.) This action has a different character from the other actions that we consider in the sequel, in that it is not faithful, except in the case where S is the torus, or in trivial case where S and is the two-sphere.
Let $H_1(S,\mathbb{R})$ be the first singular homology space of S with real coefficients. This is a real vector space of dimension $2g$. Let us fix a set of generators of $H_1(S,\mathbb{R})$ consisting of $2g$ cycles represented by oriented simple closed curves $C_1,\ldots,C_g$ and $D_1,\ldots,D_g$ whose union is topologically conjugate to the system of curves represented in Figure 3.

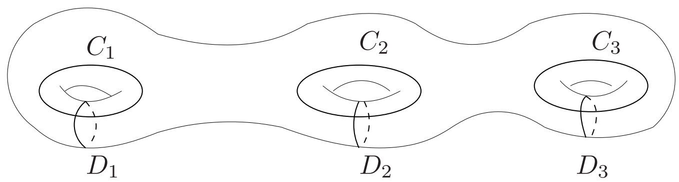

FIGURE 3. The curves shown in this picture, together with an orientation on each of them, form a basis of the real homology vector space of the surface.

If C and C' are oriented simple closed curves on S, then the *algebraic intersection* $\langle C,C'\rangle$ is defined as the sum over the intersection points $C\cap C'$ of the *signs* at these points, the sign at an intersection point being $+1$ if the orientation on C followed by the orientation on C' coincides with the orientation of the surface, and to -1 otherwise.
The algebraic intersection function $\langle C,C'\rangle$ between oriented curves extends linearly to an algebraic intersection form $H_1(S,\mathbb{R})\times H_1(S,\mathbb{R})\to\mathbb{R}$ which is skew-symmetric and non-degenerate, and which is therefore a symplectic form.
The algebraic intersection on pairs of curves in our system of generators satisfies the following:

(1) $\langle C_i,C_j\rangle=0$ for all i and j;
(2) $\langle D_i,D_j\rangle=0$ for all i and j;
(3) $\langle C_i,D_i\rangle=0$ for all i;
(4) $\langle C_i,D_j\rangle=0$ for all $i\neq j$.

Properties (1) to (4) express the fact that our basis is a symplectic basis.
There is a natural linear action of the mapping class group $\Gamma(S)$ on the vector space $H_1(S,\mathbb{R})$, induced from the action of homeomorphisms on singular cycles. This action produces a homomorphism from the mapping class group $\Gamma(S)$ to the group of linear automorphisms of the vector space $H_1(S,\mathbb{R})$.
A linear automorphism of $H_1(S,\mathbb{R})$ which is the image of a mapping class has two general features. First, it preserves the integer lattice $H_1(S,\mathbb{Z})$, that is, the subset of the vector space $H_1(S,\mathbb{R})$ consisting of the elements that can be written as linear combinations with integral coefficients of our generators, that is, as $\sum_{i=1}^{g} p_iC_i+q_iD_i$ with p_i and q_i in $\mathbb{Z}$. Indeed, such

elements in $H_1(S,\mathbb{R})$ are the singular cycles that can be represented by (not necessarily connected) oriented simple closed curves on S, and a homeomorphism of S sends an oriented simple closed curve to an oriented simple closed curve. Second, the action of an element of the mapping class group on $H_1(S,\mathbb{R})$ preserves the symplectic pairing, because an orientation-preserving homeomorphism of the surface preserves the algebraic intersection number of oriented simple closed curves. In other words, with the choice of basis that we made for $H_1(S,\mathbb{R})$, the elements of the mapping class group act as symplectic automorphisms on the lattice $H_1(S,\mathbb{Z})$. Thus, the image of $\Gamma(S)$ in the linear automorphisms group of $H_1(S,\mathbb{R})$ is contained in the group $\mathrm{Sp}(2g,\mathbb{R})$ of symplectic automorphisms of that space and, more precisely, the image is contained in the subgroup $\mathrm{Sp}(2g,\mathbb{Z})$, whose elements are called unimodular symplectic automorphisms. $\mathrm{Sp}(2g,\mathbb{Z})$ is a discrete subgroup of $\mathrm{Sp}(2g,\mathbb{R})$. Thus, we have a natural homomorphism $\Gamma(S)\to \mathrm{Sp}(2g,\mathbb{Z})$.
The following is a classical result, attributed to Burckhardt, Dehn and Nielsen.

Theorem 1.1. *For any closed surface S, the natural homomorphism*

$$\Gamma(S)\to \mathrm{Sp}(2g,\mathbb{Z})$$

is surjective.

It is easy to see that this homomorphism is not injective. For instance, a Dehn twist around a separating nontrivial simple closed curve defines a nontrivial mapping class which induces the identity automorphism on the first homology. The kernel of the homomorphism $\Gamma(S)\to \mathrm{Sp}(2g,\mathbb{Z}$ is the so-called *Torelli group*, which we shall denote by $\mathrm{Tor}(S)$. There has been a lot of activity on the Torelli group since the 1970s, and despite this fact this groups is still poorly understood. For instance, it is unknown whether it is finitely presented, for any $g\geq 3$. D. Johnson proved in [34] that the Torelli group is finitely generated for all $g\geq 3$ and G. Mess proved in [51] that for $g=2$, the Torelli group is a nonabelian free group of infinite rank and therefore is not finitely generated. Note that in the case of genus 1, the Torelli group is the trivial group.

2. Outer automorphisms of the fundamental group

Let G be a group and let $\mathrm{Aut}(G)$ be its automorphism group. Then, G acts on itself by inner automorphisms, and the image of G in $\mathrm{Aut}(G)$ is called the the group of inner automorphisms of G and is denoted by $\mathrm{Inn}(G)$. The group $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$. The *outer automorphism group of G* is the quotient group

$$\mathrm{Out}(G)=\mathrm{Aut}(G)/\mathrm{Inn}(G).$$

The study of outer automorphisms of the fundamental group of a surface was carried on by Dehn and Nielsen in the first quarter of the 20th century.

An element of $\Gamma^*(S)$ induces an automorphism of the fundamental group $\pi_1(S)$ which is only defined up to composition with an inner automorphism of $\pi_1(S)$. The ambiguity here has to do with the choice of a basepoint for the definition of $\pi_1(S)$ and with the fact that a homeomorphism of S does not necessarily preserve the basepoint. More precisely, if x is a basepoint of S, then, a continuous map $f : S \to S$ gives rise to a homomorphism $f_\sharp : \pi_1(S,x) \to \pi_1(S, f(x))$ (and not from $\pi_1(S,x)$ to itself). By composing f with a homeomorphism of S which is isotopic to the identity and which sends $f(x)$ to x, we get a homeomorphism which is isotopic to f and which sends x to itself, and which therefore defines a homomorphism of $\pi_1(S,x)$, but due to these choices this homomorphism is not canonical. Still, any element of $\Gamma^*(S)$ gives a well defined element of the outer automorphism group $\mathrm{Out}(\pi_1(S))$. In fact, one gets a natural homomorphism $\Gamma^*(S) \to \mathrm{Out}(\pi_1(S))$, and we have the following:

Theorem 2.1 (Dehn-Nielsen, Baer)**.** *For any closed surface S of genus ≥ 1, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Out}(\pi_1(S))$$

is an isomorphism.

The theorem was stated by Dehn, but it is generally considered that there were gaps in the first proof that Dehn gave.[2] Nielsen later on wrote a proof of the surjectivity [53], which he attributed to Dehn, and Baer wrote a proof of the injectivity [2].

Let us note that the proofs that Dehn and Nielsen gave of this result use hyperbolic geometry, namely, the action of mapping classes on the geodesic representatives of the elements of the fundamental group, after the surface has been equipped with a hyperbolic structure.

Note that in the case of genus 0, the theorem is false, because the extended mapping class group of the sphere is $\mathbb{Z}_2$ (generated by the class of any orientation-reversing involution) whereas the fundamental group of the sphere is trivial.

At the level of the mapping class group, we have an isomorphism

$$\Gamma(S) \to \mathrm{Out}_+(\pi_1(S))$$

where $\mathrm{Out}_+(\pi_1(S))$ is the subgroup of "orientation preserving outer automorphisms", a notion that is defined using group cohomology.[3] We note that one can also define orientation preserving automorphisms of the fundamental group by looking at the sign of the determinant of the induced action on the first homology group, after the choice of a basis.

In relation to the outer automorphism group of $\pi_1(S)$, let us note the following relatively recent "non-realization" result by Ivanov and McCarthy:

[2] It seems that Dehn wrote other proofs of that theorem, but these proofs are lost, see the commentary by John Stillwell in [8] p. 363.

[3] $\mathrm{Out}_+(\pi_1(S))$ can be defined as the subgroup of $\mathrm{Out}(\pi_1(S))$ consisting of elements that act trivially on $H^2(\pi_1(S),\mathbb{Z})$, see for instance the exposition in [52]).

Theorem 2.2. *For any closed surface S, there is no injective homomorphism* $\mathrm{Out}(\pi_1(S)) \to \mathrm{Aut}(\pi_1(S))$. *In particular, the natural quotient homomorphism* $\mathrm{Aut}(\pi_1(S)) \to \mathrm{Out}(\pi_1(S))$ *is not split.*

Ivanov and McCarthy deduce this theorem from the main result of their paper [33], which concerns injective homomorphisms between extended mapping class groups, and which we shall further refer to below (see Theorem 3.5).

Example 2.3. We consider the special case where $S = S_1$ is the torus. In this case, $\pi_1(S)$ is the abelian group $\mathbb{Z} \oplus \mathbb{Z}$, and its inner automorphism group is trivial. Thus, $\mathrm{Out}(\pi_1(S)) \simeq \mathrm{Aut}(\pi_1(S)) = \mathrm{Aut}(\mathbb{Z} \oplus \mathbb{Z})$ which is isomorphic to the group of matrices

$$\mathrm{GL}(2,\mathbb{Z}) = \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad-bc = \pm 1 \Big\}.$$

With the identification $S_1 = \mathbb{R}^2/\mathbb{Z}^2$, the surjectivity of the homomorphism $\Gamma^*(S_1) \to \mathrm{Out}(\pi_1(S_1))$ follows from the fact that any element of $\mathrm{SL}(2,\mathbb{Z})$ defines a linear map of the plane $\mathbb{R}^2$ which preserves the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ and descends to a linear map of the quotient $\mathbb{R}^2/\mathbb{Z}^2$.
A matrix in $\mathrm{GL}(2,\mathbb{Z})$ represents a homeomorphism of the torus that preserves the orientation if and only if its determinant is 1, that is, if this matrix is in the special linear group $\mathrm{SL}(2,\mathbb{Z})$.
In particular, $\Gamma^*(S_1) \simeq \mathrm{GL}(2,\mathbb{Z})$ and $\Gamma(S_1) \simeq \mathrm{SL}(2,\mathbb{Z})$.

Remark 2.4. If S is not a closed surface, then the homomorphism $\Gamma^*(S) \to \mathrm{Out}(\pi_1(S))$ is not an isomorphism, unless S is a disk or an annulus. If S is not a disk or an annulus, the homomorphism is still injective, and the image of $\Gamma^*(S)$ in $\mathrm{Out}(\pi_1(S))$ is the subgroup consisting of elements which "preserve the peripheral structure of the surface", i.e. which preserve the collection of conjugacy classes of simple closed curves that are homotopic to the punctures. This result is due to Zieschang (injectivity) and to Magnus (surjectivity onto the subgroup consisting of elements that preserve the peripheral structure), see [72].
In the case of surfaces with boundary, the statement of the realization theorem of the extended mapping class group as a group of outer automorphisms is not as satisfying as in the case of closed surfaces, because the notion of "preserving the peripheral structure" is not an instrinsic notion of the fundamental group (which in the case of a surface with boundary is a free group), but depends on the realization of that group as the fundamental group of the surface with boundary. Note however that there are group-theoretic formulations of the notion of preserving the peripheral structure, which are done in the setting of a generalization of the Dehn-Nielsen Theorem in the theory of Fuchsian groups. In this setting, one asks that parabolic elements of the Funchsian group are sent to parabolic elements. We refer the reader to [41].

3. Automorphisms of mapping class groups

As it is the case of any group, the group $\Gamma^*(S)$ acts on itself by conjugation. In [29], N. Ivanov announced the following result:

Theorem 3.1. *Let $S = S_g$ be a closed surface of genus $g \geq 3$. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(\Gamma^*(S))$$

in which Γ^* acts on itself by inner automorphisms is an isomorphism.

The result says in particular that for $g \geq 3$, any automorphism of $\Gamma^*(S_g)$ is inner. Equivalently, the outer automorphism group of $\Gamma^*(S_g)$ is trivial.
We note that this result fails in the case of a closed surface genus 2 (see below).
The following theorem is proved in the paper [48] by J. McCarthy, which also contains a precise description of the groups $\mathrm{Out}(\Gamma(S_g))$ and $\mathrm{Out}(\Gamma^*(S_g))$ for all $g \geq 2$.

Theorem 3.2. *Let $S = S_g$ be a closed surface of genus $g \geq 2$. Then,*

(1) $\mathrm{Out}(\Gamma^*(S_2)) \simeq \mathrm{Out}(\Gamma(S_2)) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$*;*
(2) for $g \geq 3$, $\mathrm{Out}(\Gamma(S_g)) = \mathbb{Z}_2$ *and* $\mathrm{Out}(\Gamma^*(S_g)) = \{1\}$.

This theorem (without the distinction between genus 2 and genus ≥ 3) was conjectured in 1983 by V. Turaev, cf. [30] p. 201.
Note that since $\Gamma(S)$ is a normal subgroup of $\Gamma^*(S)$ and that $\mathrm{Inn}(\Gamma^*(S))$ restricts to a subgroup of $\mathrm{Aut}(\Gamma(S))$.

Remark 3.3. The case of the closed surface of genus two, as analyzed by McCarthy, involves the so-called *exceptional automorphism* $\tau : \Gamma(S) \to \Gamma(S)$ which is an automorphism that maps a Dehn twist along a non-separating curve on S to its product with the unique nontrivial element of the center of $\Gamma(S)$. McCarthy proved that every automorphism of $\Gamma(S_2)$ is the restriction of an inner automorphism of $\Gamma^*(S_2)$ or the composition of such an automorphism with τ (which gives in particular $\mathrm{Out}(\Gamma(S_2)) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$).

McCarthy's proof of Theorem 3.2 is based on an analysis of the action of an automorphism of the extended mapping class group on the collection of abelian subgroups of that group. It uses the fact that an automorphism of the extended mapping class group must preserve an invariant of abelian subgroups, called *stable rank*. We note by the way that geometric invariants of abelian subgroups of mapping class groups have been studied by Birman, Lubotzky and McCarthy in [5], and that this work is based on Thurston's classification theorem of mapping classes [69].
Ivanov gave later on a general version of the rigidity result which is valid in the case of surfaces with boundary. In [30], he proved the following

Theorem 3.4. *If S is not a sphere with at most four holes or a torus with at most two holes or a closed surface of genus two, then, any automorphism* *of* $\Gamma(S)$ *is the restriction of an inner automorphism of* $\Gamma^*(S)$*. Furthermore, we have* $\mathrm{Out}(\Gamma(S)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* $\mathrm{Out}(\Gamma^*(S)) = \{1\}$*.*

Ivanov's proof is based on a clever algebraic characterization of Dehn twists in the mapping class group, showing that any automorphism of the extended mapping class group sends a Dehn twist to a Dehn twist, and moreover, preserving some geometric relations between Dehn twists (for instance, commuting twists are sent to commuting twists). Such automorphisms are then shown to be induced by homeomorphisms of the surface.

We note that Theorem 3.1 was already known in the case of braid groups, that is, mapping class groups of spheres with holes. The result is due to Dyer and Grossmann [9].

We also note that another proof of Theorem 3.1 was given by R. Tchangang in [64]. This proof is based on the analysis of the action of an automorphism of $\Gamma^*(S)$ on finite order elements of that group, and it uses the result of Dyer and Grossmann for braid groups that we stated above.

Finally, we note that Ivanov and McCarthy later on improved these results, in a study of injective homomorphisms of mapping class groups. They prove in [33] the following strong rigidity result:

Theorem 3.5. *Let* $S = S_{g,n}$ *and* $S' = S_{g',n'}$ *be two surfaces, such that* $g \geq 2$ *and* $(g', n') \neq (2, 0)$*, such that the values* $3g-3+n$ *and* $3g'-3+n'$ *differ by at most one. Then, any injective homomorphism* $\Gamma(S) \to \Gamma(S')$ *is an inner automorphism.*

The quantity $3g - 3 + n$ that appears in the statement of Theorem 3.5 is used in the paper [33] as being the *maximal rank of an abelian subgroup* of the mapping class group $\Gamma(S_{g,n})$, which is also the maximal cardinality of a set of disjoint and pairwise non-homotopic simple closed curves on $S_{g,n}$ which are not homotopic to a point or to a boundary component.

4. Automorphisms of the Torelli group

In this section, $S = S_g$ is a closed surface of genus g.

Recall from Section 1 that the Torelli group $\mathrm{Tor} = \mathrm{Tor}(S)$ is the kernel of the natural homomorphism $\Gamma(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$, arising from the action of homeomorphisms of S on homology.

The extended mapping class group $\Gamma^*(S)$ acts on the Torelli group by conjugation: if γ is an extended mapping class and η an element of the Torelli group, then the extended mapping class $\gamma\eta\gamma^{-1}$ is an element of the Torelli group. (That $\gamma\eta\gamma^{-1} \in \mathrm{Tor}(S)$ for γ in $\Gamma(S)$ is just the fact that $\mathrm{Tor}(S)$ is a normal subgroup of $\Gamma(S)$.) This action defines a homomorphism $\Gamma^*(S) \to \mathrm{Aut}(\mathrm{Tor}(S))$.

The following theorem was proved by Farb and Ivanov for $g \geq 5$ [15]. McCarthy and Vautaw gave a proof which is valid for all $g \geq 3$ [50].

Theorem 4.1 (Farb-Ivanov, McCarthy-Vautaw)**.** *Let S be a closed surface of genus $g \geq 3$. Then, the homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(\mathrm{Tor}(S))$$

defined by the action of $\Gamma^*(S)$ *on* $\mathrm{Tor}(S)$ *by conjugation is an isomorphism.*

Note that this result is false for $g = 2$, since by a result of Mess, $\mathrm{Tor}(S_2)$ is a free abelian group on an infinite set of generators (see [51]). For $g \leq 1$, the Torelli group is the trivial group, so the theorem in that case does not have content.

The proof of Theorem 4.1 that McCathy and Vautaw gave involves the action of the automorphisms of $\mathrm{Tor}(S)$ on the curve complex $C(S)$ described in Section 6 below. It uses the fact that any automorphism of $C(S)$ is induced by an element of the extended mapping class group (Theorem 6.3 below).

Note that Theorem 4.1 does not imply that the outer automorphism group of the Torelli group of a closed surface of genus $g \geq 3$ is trivial, since conjugation in that group by an element of the extended mapping class group is not an inner automorphism. Rather, the theorem implies that the outer automorphism group of the Torelli group is a degree-two extension of the integral symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$.

Part 2. ACTIONS ON SIMPLICIAL COMPLEXES AND OTHER CELL COMPLEXES

In this part, we shall consider actions of the extended mapping class group on several simplicial complexes. We note right away that there are actions of the extended mapping class group onother simplicial complexes which have been studied by various authors and which I do not mention here.

We start with a short survey on abstract simplicial complexes.

5. Abstract simplicial complexes

In this section, we review some notions related to abstract simplicial complexes which we shall use in the rest of this part. Of course, a reader familiar with the bases of abstract simplicial complexes can skip this section. More material on this subject is contained in [49].

Definition 5.1 (Simplicial complex)**.** An *abstract simplicial complex K with vertex set V* is a (finite or infinite) set V equipped with the structure defined by a collection of finite subsets of V such that

(1) $x \in V$ implies $\{x\} \in K$;
(2) $\sigma \in K$ and $\tau \subset \sigma$ implies $\tau \in K$.

In what follows, all simplicial complexes will be abstract, and therefore we shall omit this adjective.

The elements of the set V are called the *vertices* of K.

We shall sometimes talk about the simplicial complex V (with its structure of subsets being understood), instead of the simplicial complex K.

A finite subset of V belonging to this structure is called a *simplex* of the simplicial complex. The *dimension* of a simplex σ is equal to $\mathrm{Card}(\sigma)-1$. A simplex of dimension k is called a k-simplex. In particular, the 0-simplices of K are the vertices of K. A simplex of dimension 1 is called an *edge*.
The *dimension* of the simplicial complex K, $\dim(K) \in \mathbb{N} \cup \{\infty\}$, is equal to the maximal dimension of a simplex in K.

Example 5.2. If V is an arbitrary set (finite, infinite, countable or uncountable), then the set of all finite subsets of V has a natural structure of a simplicial complex.

A simplicial complex is said to be *locally finite* if every vertex belongs to only finitely many simplices.
An abstract simplicial complex K is said to be *connected* if we can join any two vertices by a sequence of consecutive edges, i.e. if for any vertices v and w of K, there exists a sequence of edges $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\ldots, \{v_{n-1}, v_n\}$, with $v_0 = v$ and $v_n = w$.

Definition 5.3. A *subcomplex* of a simplicial complex K is a simplicial complex H whose vertex set is contained in the vertex set of K and such that any simplex of H is a simplex of K.

Note that there could be simplices of K which are not simplices of H even though they have all their vertices in H.

Example 5.4. 1) Let K be a simplicial complex and let σ be a simplex of K. Then, the set of all subsets of σ has a natural structure of a simplicial complex whose vertex set is the set of vertices of σ, and it is a subcomplex of K.
2) Let K be a simplicial complex. For each $n \geq 0$, the *n-skeleton* K_n of K is the set of all k-simplices of K with $k \leq n$. This set has a natural structure of a simplicial complex, with the same vertex set than K, and it is a subcomplex of K.

Definition 5.5 (Flag complex)**.** A simplicial complex K is called a *flag complex* if for every set $\{x_0, \ldots, x_n\}$ of vertices of K such that for all i and j satisfying $0 \leq i < j \leq n$, $\{x_i, x_j\}$ is an edge of K, the set $\{x_0, \ldots, x_n\}$ is a simplex of K.

Thus, as a simplicial complex, a flag complex is completely determined by its one-skeleton. Note however that despite this fact, there are important issues concerning flag complexes that do not only depend on the one-skeleton. For instance the automorphism group of a flag complex is generally not equal to that of its one-dimensional skeleton.

Definition 5.6. Let K be a simplicial complex with vertex set V and let K' a simplicial complex with vertex set V'. A *simplicial map* from K to K' is a map $f_0 : V \to V'$ whose induced action on subsets, denoted by $f : K \to K'$, takes a simplex of K to a simplex of K'. The map f_0 is then the induced map from the simplicial map $f : K \to K$ on the set of vertices of K and K'.

If we denote our two simplicial complexes by V and V' (with the structures K and K' of subsets being understood), then a simplicial map between these complexes can also be denoted by $f : V \to V'$, but one has to be careful on the fact that this map does not only denote the induced map on vertices.
If the simplicial map $f : K \to K$ is bijective, then f is said to be a *simplicial isomorphism*. (In particular, the induced map on the vertices, $f_0 : V \to V'$, must be bijective.)

Example 5.7. If V and V' are two sets and if K and K' are the simplicial complexes whose simplices are the sets of all finite subsets of V and V' respectively, then, any map $f_0 : V \to V'$ between the sets of vertices of K and K' induces a simplicial map $f : K \to K'$. If f_0 is a bijection, then f is an isomorphism.

If $f : K \to K'$ is a simplicial map which is injective, then we can identify K with a subset H of K'. This subset H inherits the structure of a simplicial complex, by pushing forward the structure of the simplicial complex K. Note that H is a subcomplex of K'.
If $f : K \to K'$ is a simplicial map from a simplicial complex K to a simplicial complex K', and if $g : K' \to K$" is a simplicial map from the simplicial complex K' to a simplicial complex K'', then $g \circ f$ is a simplicial map from the simplicial complex K to the simplicial complex K''.

Definition 5.8 (the star of a vertex of a simplicial complex)**.** Let x be a vertex of a simplicial complex K. The *star of x in K* is the subcomplex $\mathrm{St}(x, K)$ of K whose simplices are the simplices of K which contain the vertex x together with all the faces of such simplices of K.

Let K be a simplicial complex and x be a vertex of K. Note that the 0-skeleton $\mathrm{St}_0(x, K)$ of $\mathrm{St}(x, K)$ is the set of all vertices w of K such that $\{x, w\}$ is a simplex of K.
The following proposition is useful in the study of flag complexes:

Proposition 5.9. *Let K be a flag complex. Let x and y be vertices of K. Then the following are equivalent:*

(1) $\mathrm{St}(x, K) = \mathrm{St}(y, K)$.
(2) $\mathrm{St}_0(x, K) = \mathrm{St}_0(y, K)$.

Definition 5.10 (The link of a vertex of a simplicial complex)**.** Let x be a vertex of a simplicial complex K. The *link of x in K* is the subcomplex $\mathrm{Lk}(x, K)$ of K whose simplices are the simplices of $\mathrm{St}(x, K)$ which do not have x as a vertex.

We end this section by recalling the following notion that we shall use in later sections.

Definition 5.11 (Induced subcomplex)**.** Let K be a simplicial complex with vertex set V and W be a subset of V. Let K_W be the set of all simplices of K which have all their vertices in W. Then K_W is a subcomplex of K

with vertex set W, which is called the *subcomplex of K induced by the subset $W \subset V$*.

6. Automorphisms of the curve complex

We consider again a surface $S = S_{g,n}$ of genus $g \geq 0$ and with $b \geq 0$ boundary components.
A *curve* on S is a connected one-dimensional submanifold in the interior of S. A curve is said to be *essential* if it is not homotopic to a point or to a boundary component of S. For instance, there are no essential curves on a sphere with at most three holes.

Definition 6.1. The *curve complex* $C(S)$ is the flag simplicial complex whose k-simplices, for all $k \geq 0$, are the sets whose elements are $k+1$ isotopy classes of essential pairwise non-isotopic and pairwise disjoint essential curves.

The curve complex was introduced by Harvey in 1978, with the idea that this complex encodes some boundary structure of Teichmüller space, in analogy to Tits buildings which encode a boundary structure of symmetric spaces. This complex turned out to be an extremely interesting object, and it has been studied for itself by Ivanov, Masur, Minsky, Hammenstaedt, Bowditch and others.
The extended mapping class group acts simplicially on $C(S)$ in the following natural manner: if $\gamma \in \Gamma^*(S)$ is the class of a homemorphism f of S, and if σ is a simplex of $C(S)$, which is represented by a collection of curves $C_1, \ldots, C_k$, then $\gamma(\sigma)$ is the simplex represented by the collection of curves $f(C_1), \ldots, f(C_k)$.
Note that a finite collection of vertices of $C(S)$ forms a simplex of $C(S)$ if and only if each pair of vertices in this collection can be represented by disjoint curves on S. Then, $C(S)$ is a flag complex.
The complex $C(S)$ is empty when S is a sphere with at most three holes, and $C(S)$ is an infinite set of vertices when S is a sphere with four holes or a torus with at most one hole.
If S is not a sphere with at most four holes or a torus with at most one hole, then $C(S)$ is connected. This result was stated by Harvey in [21]. Proofs were given by Harer in [20] and by Masur and Minsky in [46]. The proof that Masur and Minsky gave in [46] (Lemma 2.1) uses induction on the number of intersection points between curves. In fact, Masur and Minsky gave an upper bound of the distance between two vertices in terms of the intersection number of the curves that represent these vertices. Ivanov gave in [32] another proof of the same fact that uses Cerf theory.
The complex $C(S)$ is always finite-dimensional. Indeed, there is an upper bound for the number of pairwise disjoint and non-isotopic essential curves on S. The least such upper bound is $3g-3+n$ (which is equal to the number of essential curves in a pants decomposition of S). Therefore, the dimension

of $C(S)$ is equal to $3g-4+n$. Note that this dimension is ≥ 1 provided S is not a sphere with at most four holes and a torus with at most one hole.
The complex $C(S)$ is not locally finite, provided it is connected. The reason is that as soon as a surface contains an essential curve, it contains infinitely many such curves. Thus, if α is an essential curve on S and if $C(S)$ is connected, then there are infinitely many distinct essential curves on the surface S obtained from S by cutting it along α, and therefore the vertex representing α in $C(S)$ belongs to infinitely many edges.
Although we did not define the geometric realization of a simplicial complex, we note the following result of J. Harer: the geometric realization of $C(S)$ is $(2g+n-4)$ connected if $n>0$ and $(2g-3)$ connected if $n=0$ (cf. [20] p. 217). Hare proved also that in the case where S is not a closed surface, the geometric realization of $C(S)$ is homotopy equivalent to a wedge of spheres of dimension $2g-3+n$.
The extended mapping class group $\Gamma^*(S)$ acts naturally on $C(S)$, and we thus obtain a natural homomorphism $\Gamma^*(S)$ into the group $\mathrm{Aut}(C(S)$ of simplicial automorphisms of $C(S)$.
The basic result on the automorphism group of $C(S)$ is due to N. Ivanov, who proved the following (see [30]):

Theorem 6.2. *For $g\geq 2$, the natural homomorphism*

$$\Gamma^*(S)\to \mathrm{Aut}(C(S)$$

is an isomorphism provided S is not the closed surface of genus 2. If S is the closed surface of genus 2, then the homomorphism is surjective and its kernel is $\mathbb{Z}_2$, generated by the hyperelliptic involution.

Korkmaz continued the analysis made by Ivanov and he studied the case of surfaces of genus 0 and 1. He proved in [36] that for such surfaces, any automorphism of $C(S)$ is induced by an element of $\Gamma^*(S)$ if S is not a sphere with ≤ 4 holes or a torus with ≤ 2 holes.
In the cases where $S=S_{1,1}$ and $S_{0,4}$, there are automorphisms of $C(S)$ that are not geometric since in each of these cases the complex of curves is an infinite countable set of vertices, and therefore its automorphism group is uncountable.
Luo in [39] analyzed a delicate remaining case, which is the case where the surface S is a torus with two holes. He proved that in that case the map $\Gamma^*(S)\to \mathrm{Aut}(C(S)$ is not surjective. More precisely, Luo noticed that there is an isomorphism $C(S_{1,2})\to C(S_{0,5})$. This isomorphism is induced by the projection map $\pi : S_{1,2}\to S_{1,2}/\iota$, where ι is the hyperelliptic involution of $S_{1,2}$, and where $S_{0,5}$ is identified with the complement of the singular locus of π in $S_{1,2}/\iota$. Thus, the automorphism group of $C(S_{1,2})$ is isomorphic to the automorphism group of $C(S_{0,5})$. Now it is known that the extended mapping class groups $\Gamma^*(S_{1,2})$ and $\Gamma^*(S_{0,5})$ are not isomorphic. (More precisely, $\Gamma^*(S_{1,2})$ is a subgroup of index 5 in $\Gamma^*(S_{0,5})$.) Thus, we have $\Gamma^*(S_{1,2})\not\simeq \mathrm{Aut}(S_{1,2})$.

The homomorphism $\Gamma^*(S_{1,2}) \to \mathrm{Aut}(C(S_{1,2})$ is also not injective, since the hyperelliptic involution ι acts trivially on $C(S_{1,2})$. (This was already known, from works of Birman and of Viro, cf. [4] and [70].)
In the following theorem, we summarize the results on the automorphism group of the complex $C(S)$.

Theorem 6.3 (Ivanov-Korkmaz-Luo). *Consider a surface $S_{g,n}$ whose curve complex $C(S)$ has dimension ≥ 1. (Equivalently, the curve complex of $C(S)$ is connected; equivalently, S is not a sphere with at most four holes or a torus with at most one hole). Then,we have the following.*

(1) For $(g,n) \notin \{(1,2),(2,0)\}$, the natural homomorphism

$$\Gamma^*(S_{g,n}) \to \mathrm{Aut}(C(S_{g,n}))$$

is an isomorphism.

(2) The homomorphism $\Gamma^*(S_{2,0}) \to \mathrm{Aut}(C(S_{2,0}))$ is surjective and its kernel is of order two, generated by a hyperelliptic involution.

(3) The homomorphism $\Gamma^*(S_{1,2}) \to \mathrm{Aut}(C(S_{1,2}))$ is neither surjective nor injective. The kernel of this homomorphism is of order two, generated by the hyperelliptic involution, and its image is a subgroup of index 5 in $\mathrm{Aut}(C(S_{1,2}))$. The image consists in the simplicial automorphisms of $(C(S_{1,2})$ that preserve the set of vertices represented by nonseparating curves.

Remark 6.4. The theorem of Ivanov-Kormkaz and Luo, as it is usually stated, also includes a discussion of the case where the surface is a once-punctured torus or a four-holed sphere. In these cases, the definition of the vertex set of the complex of curves is modified, so that this complex becomes one-dimensional and connected, and the study of its simplicial automorphism group becomes an interesting question. (We already noted that with the usual definition, the complex is a countable union of vertices and, therefore, its automorphism group is the group of permutations of an infinte set, which is uncountable.) The modified definition is done by requiring the following:
• In the case of the torus with one hole, two vertices of the curve complex form an edge if and only if they can be represented by simple closed curves intersecting in exactly one point. With such a definition, the natural homomorphism $\Gamma^*(S_{1,1}) \to \mathrm{Aut}(C(S_{1,1}))$ is surjective and its kernel is of order two, generated by a hyperelliptic involution.
• In the case of the sphere with four holes, two vertices of the curve complex form an edge if and only if they can be represented by simple closed curves in minimal intersection position intersecting in exactly two points. With this definition, the homomorphism $\Gamma^*(S_{0,4}) \to \mathrm{Aut}(C(S_{0,4}))$ is surjective and its kernel is $\mathbb{Z}_2 \times \mathbb{Z}_2$, generated by two hyperelliptic involutions.

We note finally that Luo, in his paper [39], gave a proof of Thorem 6.3 which includes all the cases and which is different from the proofs by Ivanov and by

Korkmaz. Luo's proof uses induction, and it is in the spirit of Grothendieck's reconstruction principle.

7. Automorphisms of the pants decomposition complex

We first recall a few definitions.
A *pants decomposition* of S is a maximal collection of essential curves $\{C_i\}$ on S which are pairwise disjoint and nonhomotopic. Equivalently, each connected component of $S \setminus \{C_i\}$ is a sphere with three holes (i.e. a pair of pants). Equivalently, the set $\{C_i\}$ represents a top-dimensional simplex in the curve complex $C(S)$.
The cardinality of a pants decomposition of S is equal to $3g-3+n$.
We shall also need the following notion:
A *simple move* between two pants decompositions is a transformation in which a single curve is modified (that is to say, the two pants decompositions involved in that move contain the same set of curves except for one curve) such that this curve and its image by the move have the smallest possible intersection number. This number is equal to one or to two, depending on whether the curve that is transformed is on the boundary of one or of two pairs of pants in each decomposition. The two types of simple moves are represented in Figure 4.

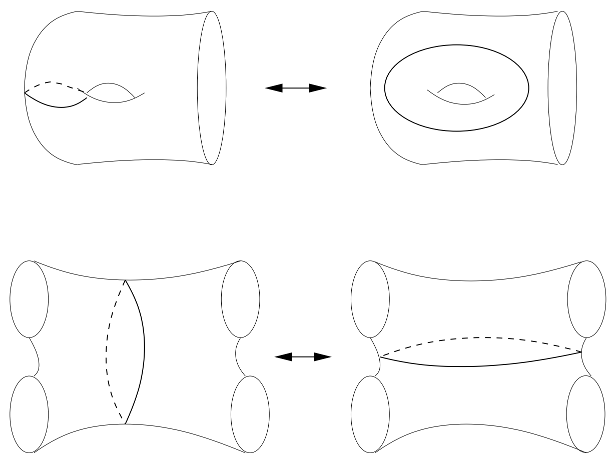

Figure 4. The two types of simple moves that define the edges of the pants decomposition graph.

We regard a simple move as an operation which is well defined up to isotopy, so that we can talk of two isotopy classes of pants decomposition that are obtained from each other by a simple move.

Definition 7.1 (Pants decomposition graph)**.** The *pants decomposition graph* $P_1(S)$ is the one-dimensional simplicial complex whose vertices are isotopy classes of pants decompositions and where two vertices are joined by an edge if the two pants decompositions that represent them (up to homotopy) differ by an simple move.

The pants decomposition graph was introduced by Hatcher and Thurston in the appendix to their paper [24]. Hatcher and Thurston proved that the pants decomposition graph $P_1(S)$ is connected, that is, any two isotopy classes of pants decompositions on a surface can be obtained from each other by a finite sequence of simple moves.
In the paper [22], Hatcher studied relations between simple moves. He highlighted five types of relations, and he constructed a two-dimensional cell complex whose 1-skeleton is the graph complex and whose 2-cells are attached via these five types of relations. These relations are better described in pictures rather than in words. The first first two types of relations are called *triangle* relations, and they are represented in Figure 5.

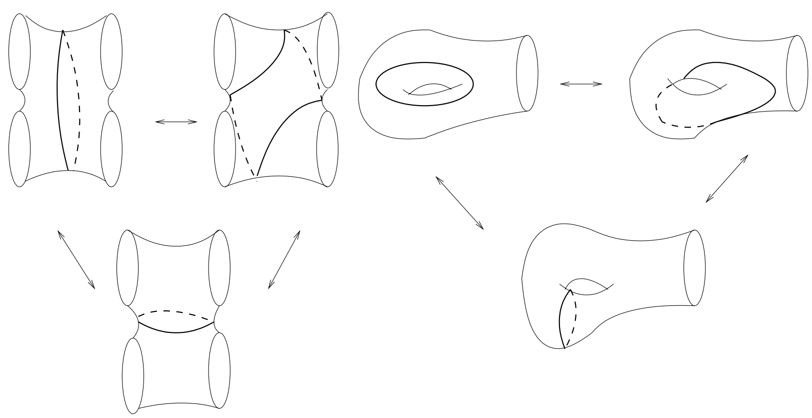

Figure 5. The two types of triangle relations in the pants decomposition graph.

The third type of relation is called a *pentagon* relation, and it is represented in Figure 6.
The fourth type of relation is called a *hexagon* relation, and it is represented in Figure 7.
For the relation of the fifth type, one starts by noting that two simple moves (where each move can be of either of the two types represented in Figure 4) which are supported on disjoint subsurfaces of S commute. Their commutator represents then a cycle of four moves, which, by definition, is a relation of the fifth type, and which is called a *quadrilateral* relation. An example of such a relation is represented in Figure 8.
In the paper [22], Hatcher made the following definition:

Definition 7.2 (Pants decomposition complex)**.** The *pants decomposition complex* $P(S)$ is the two-dimensional cell complex obtained by attaching to the pants decomposition graph all the two-dimensional cells that represent the five types of relations described above.

Hatcher proved that the two-dimensinal pants decomposition complex is simply connected.

Note that the pants decomposition graph (and therefore the pants decomposition complex) is empty if the surface S has nonnegative Euler characteristic, since in this case there is no pair of pants embedded in S.

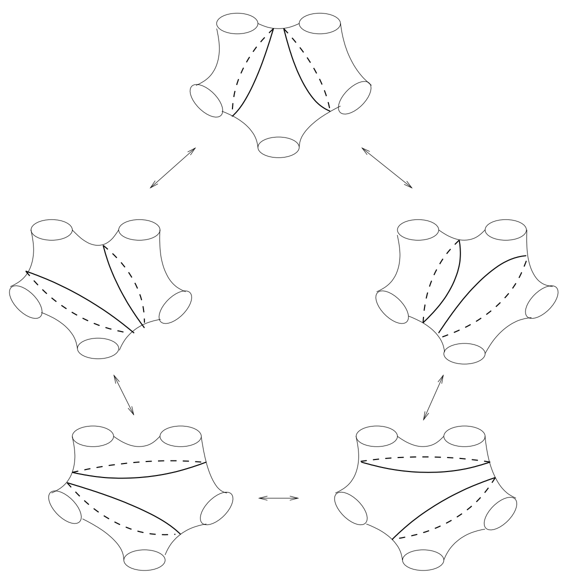

FIGURE 6. The pentagon relation in the pants decomposition graph.

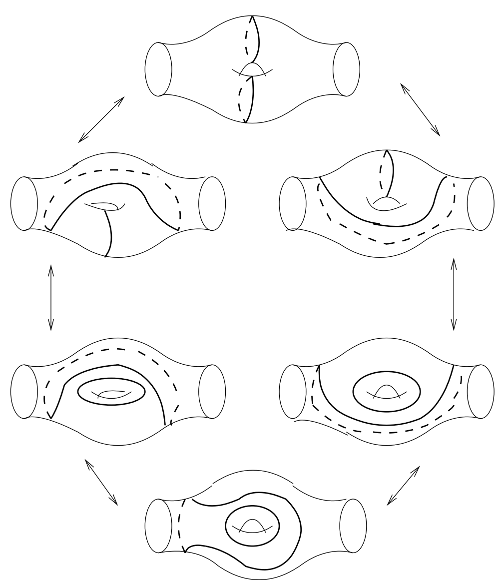

FIGURE 7. The hexagon relation in the pants decomposition graph.

The two-dimensional pants decomposition complex is also described in the paper [23] by Hatcher, Lochak and Schneps, where it is used in the context of the Grothendieck-Teichmüller theory.
Margalit proved the following rigidity result:

Theorem 7.3 (cf. [43])**.** *Let $S_{g,n}$ be a surface of negative Euler characteristic. If $(g,n) \notin \{(0,3),(1,1),(1,2),(2,0),(0,4)\}$, then the homomorphism*

$$\Gamma^*(S_{g,n}) \to \mathrm{Aut}(P(S_{g,n}))$$

is an isomorphism. In the exceptional cases, we have the following:

(1) *The homomorphism $\Gamma^*(S_{0,3}) \to \mathrm{Aut}(P(S_{0,3}))$ is not injective. (This is because $P(S_{0,3})$ is reduced to a point, and therefore $\mathrm{Aut}(P(S))$ is trivial, whereas $\Gamma^*(S_{0,3})$ is not the trivial group.)*
(2) *The homomorphism $\Gamma^*(S_{0,4}) \to \mathrm{Aut}(P(S_{0,4}))$ is surjective, and its kernel kernel is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by two hyperelliptic involutions.*
(3) *In the case where $(g,n) = (1,1),(1,2)$ or $(2,0)$, the homomorphism $\Gamma^*(S_{g,n}) \to \mathrm{Aut}(P(S_{g,n}))$ is surjective, and its kernel is $\mathbb{Z}_2$, generated by a hyperelliptic involution.*

The proof by Margalit of Theorem 7.3 uses the result of Ivanov, Korkmaz and Luo on the automorphisms of the curve complex (Theorem 6.3 above). In fact, Margalit constructs an isomorphism between the groups $\mathrm{Aut}(P_1(S))$ and $\mathrm{Aut}(C(S))$ which commutes with the natural homomorphism from of the extended mapping class group into these two groups.
We also note that in the same paper, Margalit also proves the following:

Theorem 7.4. *For any surface S of negative Euler characteristic, we have*

$$\mathrm{Aut}(P(S_{g,n})) \simeq \mathrm{Aut}(P_1(S_{g,n})).$$

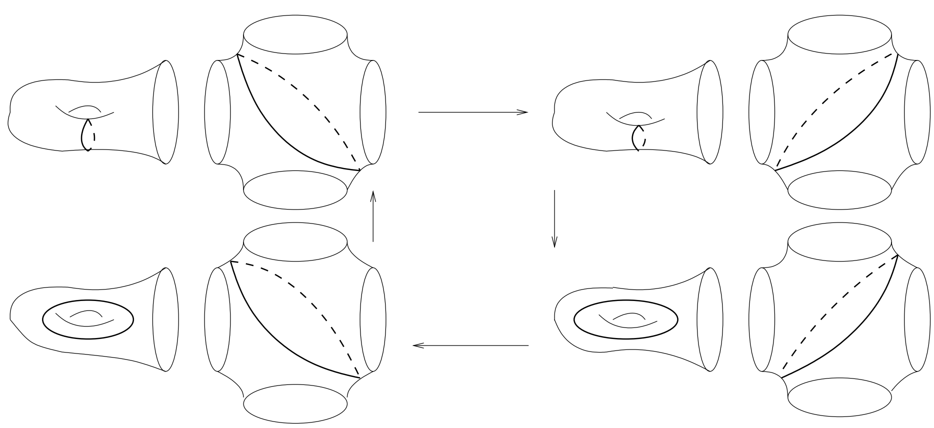

FIGURE 8. An example of a quadrilateral relation in the pants decomposition graph.

8. Automorphisms of the arc complex

An *arc* in S is the homeomorphic image of a compact interval of $\mathbb{R}$. An arc s in S is said to be *properly embedded* if $s \cap (S \setminus \partial S) = \partial s$. A properly embedded arc s is said to be *essential* if there is no closed disk embedded in S whose boundary consists of the union of s with an arc contained in ∂S.
In this section, all arcs in S are supposed to be properly embedded and essential, and we shall omit these adjectives when talking about arcs.

Definition 8.1. The *arc complex* of S is the flag simpicial complex whose k-simplices, for every $k \geq 0$, are the isotopy classes of $k+1$ pairwise non-isotopic disjoint arcs on S.

As special cases, if $S = S_{0,3}$ is a sphere with three holes, then $A(S_{0,3})$ is connected and two-dimensional, with a finite number of simplices. If $S = S_{1,1}$ is a torus with one hole, then $A(S_{1,0})$ is connected, one-dimensional and infinite (it is a Farey graph).
Note that if S is a closed surface, then $A(S)$ is empty.
Homeomorphisms and isotopes of S take disjoint arcs to disjoint arcs, and the extended mapping class group of S acts simplicially on $A(S)$.
Irmak and McCarthy gave a complete description of the automorphism group of the arc complex. Their result is the following:

Theorem 8.2 (Irmak-McCarthy [28])**.** *Let $S_{g,n}$ be a surface with nonempty boundary and with negative Euler characteristic. Then, the natural homomorphism*
$$\rho : \Gamma^*(S) \to \mathrm{Aut}(A(S))$$
is an isomorphism provided $(g,n) \notin \{(1,1),(0,3)\}$. In the exceptional cases, the kernel of ρ is the centre of $\Gamma^*(S)$. In other words, we have the following:
1. *if S is a pair of pants, the kernel of ρ is $\mathbb{Z}_2$, generated by the isotopy class of any orientation-reversing involution of S that preserves each boundary component of S;*
2. *if S is a torus with one hole, the kernel of ρ is $\mathbb{Z}_2$, generated by a hyperelliptic involution of the surface S.*

We note that Irmak and McCarthy obtained a stronger result, namely, they proved that any injective simplicial self-map of $A(S)$ is induced by a homeomorphism of S.

9. Automorphisms of the Schmutz graph of nonseparating curves

In [63], Paul Schmutz Schaller introduced and studied a new one-dimensional simplicial complex $G(S)$ associated to S. The definition depends on whether the genus of S is 0 or ≥ 1.
We recall that the *geometric intersection number* of two simple closed curves is the minimum number of intersection points of two representatives of the isotopy classes of these curves.

Definition 9.1 (The Schmutz graph)**.** Let $S = S_{g,n}$ be a surface of negative Euler characteristic which is not a pair of pants. Then:

(1) For $g \geq 1$, the vertex set of $G(S)$ is the set of isotopy classes of nonseparating simple closed curves on S, and two vertices are related by an edge whenever their geometric intersection number is 1.
(2) For $g = 0$, the vertex set of $G(S)$ is the set of isotopy classes of simple closed curves on S which separate S into two components one of which is a pair of pants. (Note that two of the boundary components of this pair of pants are boundary components of S, and therefore such a boundary component does not exist if $b \leq 1$.) In this case, two vertices are related by an edge whenever their geometric intersection is equal to two.

Schmutz Schaller proved that $G(S)$ is connected and that the automorphism group of this complex is equal to the mapping class group modulo its centre. More precisely, he proved the following:

Theorem 9.2 (Schmutz Schaller [63])**.** *Let $S = S_{g,n}$ be a surface of negative Euler characteristic which is not a pair of pants. Then, if $(g, n) \notin \{(0, 4), (1, 1), (1, 2), (2, 2)\}$, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{G}((S))$$

is an isomorphism. Furthermore, in the exceptional cases, the situation is as follows:

(1) for $(g, n) \in \{(1, 1), (1, 2), (2, 2)\}$, the homomorphism is surjective, and its kernel is $\mathbb{Z}_2$, generated by a hyperelliptic involution of S;
(2) for $(g, n) = (0, 4)$, the homomorphism is surjective, and its kernel is is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by two hyperelliptic involutions.

In his study of the complex $G(S)$, Schmutz Schaller was motivated by the study of the structure of the collection of systoles of hyperbolic surfaces. We note that in this context, it is more natural to think of $S_{g,n}$ as a surface of genus g with n cusps rather than a surface of genus g with n boundary components.

10. Automorphisms of the complex of nonseparating curves

Definition 10.1. The complex $N(S)$ of nonseparating curves is the flag simplicial complex whose k-simplices, for every $k \geq 0$, are the isotopy classes of nonseparating simple closed curves on S.

Note that $N(S)$ is canonically isomorphic to the subcomplex of $C(S)$ induced by the set of vertices that are isotopy classes of nonseparating simple closed curves.
E. Irmak proved the following:

Theorem 10.2 (Irmak [26]). *Suppose that the genus of S is ≥ 2. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(N(S))$$

is an isomorphism, except if S is the closed surface of genus 2, in which case the automorphism group of $N(S)$ is $\Gamma^*(S)/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the hyperelliptic involution of S.

11. Automorphisms of the systolic complex of curves

In this section, we describe another simplicial complex that appears in the study of systoles of hyperbolic surfaces, which was defined by Schmutz Schaller in his paper [63].

The definition of the systolic complex of curves depends on whether the number of boundary components of the surface is ≤ 1 or ≥ 2.

Definition 11.1. The *systolic complex of curves $SC(S)$* is the simplicial complex defined as follows:

(1) If $b \leq 1$, then the vertices of $SC(S)$ are the homotopy classes of non-separating simple closed curves on S, and for all $k \geq 1$, the k-simplices of $SC(S)$ are the collections of $k+1$ homotopy classes of non-separating simple closed curves on S which mutually intersect in at most one point.

(2) If $b \geq 2$, then the vertices of $SC(S)$ are either homotopy classes of non-separating simple closed curves on S or homotopy classes of simple closed curves which have one of their complementary components homeomorphic to a pair of pants. (Note that this implies that two of the boundary components of the pair of pants are boundary components of S.) For $k \geq 1$, the k-simplices of $SC(S)$ are sets of $k+1$ vertices such that two homotopy classes of non-separating curves representing two vertices of this simplex intersect at most once, and such that any homotopy class of separating curves representing another vertex of such a simplex intersects any other curve representing a vertex at most twice.

P. Schmutz Schaller studied the complex $SC(S)$ as a natural combinatorial object that encodes some information on the set of systoles on the surface S. In this context, as in the context of the Schmutz graph of Section 9, the surface is realized as a hyperbolic surface with cusps instead of boundary components. In see [63] p. 244, Schmutz Schaller makes the following conjecture

Conjecture 11.2. *The automorphism group of the systolic complex $SC(S)$ is isomorphic to the extended mapping class group.*

Schmutz Schaller says he can prove this conjecture in the cases where $(g, n) = (1, 1), (1, 2), (2, 0)$ and $(2, 0)$. He considers Theorem 9.2 as a step towards proving conjecture 11.2.

12. Automorphisms of the Hatcher-Thurston complex of cut systems

The Hatcher-Thurston complex of cut systems is a two-dimensional cell complex (which is not a simplicial complex). In order to define it, we need to recall the definition of a cut system.

A *cut system* on S is the isotopy class of a set of pairwise disjoint closed curves such that the surface S cut along this collection of curves is a sphere with holes. Note that the number of holes is then equal to $2g+2$.

We note that if the genus of S is 0, then there is no cut system on S. Thus, for the rest of this section, we suppose that the genus of S is ≥ 1.

Note that each of the curves defining a cut system is necessarily nonseparating, and that the genus of the surface S is (by the definition of the genus) the cardinality of a cut system on S.

Hatcher and Thurston introduced a notion of *simple move* between cut systems.

A *simple move* consists in replacing, in a cut system, one of the curves with a new curve, such that the old and the new curves intersect in exactly one point. There are two sorts of simple moves, and they are the represented by the same pictures than those that represent the simple moves of the Hatcher-Thurston pants graph (see Figure 4 above).

In the same way as for simple moves between pants decompositions, we regard a simple move between cut systems as an operation which is well defined up to isotopy, so that we can talk of two isotopy classes of cut systems that are obtained from each other by a simple move.

Definition 12.1 (The cut-system graph)**.** We suppose that the genus of $S_{g,n}$ is ≥ 1. The *cut-system graph* is the simplicial graph whose vertex set is the set of isotopy classes cut systems on S and whose edges are the pairs of cut systems that are related by a simple move.

Note that in the case where the genus of S is 1, a cut system is reduced to a single nonseparating curve, and that in this case the 1-dimensional cut-system graph coincides with the Schmutz graph $G(S)$ defined in Section 9 above.

The graph $HT_1(S)$ is the one-skeleton of a two-dimensional cell complex, which we shall call the *(2-dimensional) Hatcher-Thurston complex*, which we denote by $HT(S)$ and which we shall define below.

We shall call a *path* in a graph G a finite sequence $v_1,\dots,v_n$ of vertices of G such that any two consecutive vertices in this sequence are connected by an edge. Such a path is said to be a *cycle* if $v_1=v_n$.

Hatcher and Thurston introduced three types of cycles in the complex $HT_1(S)$, called *distinguished cycles*. These cycles define the *relations* between sequences of simple moves in the sense that if we have two different sequences of simple moves that join two given cut systems, then we can pass from one sequence to the other using distinguished cycles.

three types of distinguished cycles are the following:

- A *Triangular cycle:* This is a closed path in $HT_1(S)$ having three vertices represented by three cut systems that have $g-1$ curves in common and such that the remaining curves in these three cut-systems are represented by simple closed curves C_1, C_2, C_3 whose intersection pattern is represented in Figure 9 (a).
- A *Rectangular cycle:* This is a closed path in $HT_1(S)$ having four consecutive vertices represented by cut systems that have $g-2$ curves in common, and such that the remaining pairs of curvess in the four cut-systems are pairs $\{D_1, C_1\}$, $\{C_1, D_2\}$, $\{D_2, C_2\}$, $\{C_2, D_1\}$ whose intersection pattern is represented in Figure 9 (b). (The cyclic order given here represents the cyclic order of the path in $HT_1(S)$.)
- A *Pentagonal cycle:* This is a closed path in $HT_1(S)$ having five consecutive vertices represented by cut systems that have $g-2$ curves in common, and such that the remaining pairs of curves in these cut-systems are pairs $\{C_2, C_5\}$, $\{C_5, C_3\}$, $\{C_3, C_1\}$, $\{C_1, C_4\}$, $\{C_4, C_2\}$ whose intersection pattern is represented in Figure 9 (c). (Again, the cyclic order given here represents the cyclic order of the path in $HT_1(S)$.)

Now we can give the following

Definition 12.2 (The two-dimensional Hatcher-Thurston cell complex)**.** Let $S_{g,n}$ be a surface of genus $g \geq 1$. The *two-dimensional Hatcher-Thurston cell complex* $HT(S)$ is obtained from the one-dimensional cut system complex by adding for each distinguished closed path a two-cell with attaching map this distinguished closed path.

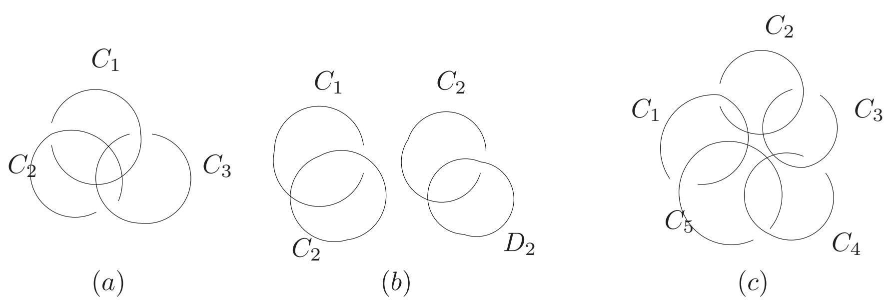

FIGURE 9. The triangle, the quadrilateral and the pentagonal cycles in the Hatcher-Thurston cell complex

Hatcher and Thurston proved in [24] that the complex $HT(S)$ is connected and simply connected provided the genus of S is at least one and they used this complex to find a presentation of the mapping class group [24]. Let us also note that Harer used this complex in his computation of the second homology group of the mapping class group [19].

The automorphisms of this complex were studied by E. Irmak and M. Korkmaz, who proved in [27] that the group $\mathrm{Aut}(HT(S))$ of cellular automorphisms of $HT(S)$ is the extended mapping class group modulo its centre. More precisely, they proved the following.

Theorem 12.3 (Irmak and Korkmaz [27]). *For any $g \geq 1$ and $n \geq 0$ satisfying $(g,n) \notin \{(1,0),(1,1),(1,2),(2,0)\}$, the natural map*

$$\Gamma^*(S_{g,n}) \to \mathrm{Aut}(HT(S_{g,n})$$

is an isomorphism. In the exceptional cases, theis map is surjective and its kernel is $\mathbb{Z}/2$.

Irmak and Korkmaz proved Theorem 12.3 using the Schmutz complex. A beautiful ingredient in their proof is the encoding of nonseparating simple closed curves on S by vertices and edges in the Hatcher-Thurston complex. (Remember that every curve in a cut-system, representing a vertex of the Hatcher-Thurston complex $HT(S)$, is nonseparating.) Using this, the authors show that any automorphism of $HT(S)$ induces an automorphism of the set of nonseparating simple closed curves. They than show that such an automorphism sends a pair of isotopy classes of nonseparating curves whose geometric intersection number is equal to one to a pair having the same property. From this, they construct a homomorphism from $\mathrm{Aut}(HT(S))$ to the Schmutz complex $G(S)$, and they finally prove that this map is an isomorphism.

We finally note that in her paper [26] p. 84, Irmak says that the Isomorphism Theorem 12.3 can also be deduced from her result about the automorphism group of the complex of nonseparating curves (Theorem 10.2 above).

13. Automorphisms of the complex of domains

In this section, I describe joint work with John McCarthy.

As before, $S = S_{g,n}$ is a compact connected orientable surface genus g with n boundary components (genus g with n holes).

We shall say that a curve α in S is *k-peripheral* if there exists a sphere with k holes embedded in S such that α is a boundary component of X and all the other boundary components of X are boundary components of S.

Note that a curve α is essential if it is not 0 or 1-peripheral.

A *domain* X in S is a compact connected subsurface with boundary of S such that

- $X \neq S$
- each component of ∂X is either a boundary component of S or an essental curve in S.

Definition 13.1 (The complex of domains). The complex of domains $D(S)$ is the flag simplicial complex whose k-simplices, for each $k \geq 0$, are the isotopy class of $k+1$ pairwise disjoint and non-isotopic domains.

The following special vertices of $D(S)$ play an important role in the theory of the complex of domains:

- *annular vertices*: isotopy classes of regular neighborhoods of essential curves.
- *vertices represented by biperipheral pairs of pants*: isotopy classes of domains homeomorphic to pairs of pants which have two boundary components that are boundary components of S.

The extended mapping class group $\Gamma^*(S)$ acts on the collection of isotopy classes of domains, sending a pair of isotopy classes that can be represented by disjoint domains to a pair of isotopy classes that have the same property. Therefore, we have a homomorphism from $\Gamma^*(S)$ to the simplicial automorphisms group $\mathrm{Aut}(D(S))$. For instance, one can use the action induced by $\Gamma^*(S$ on the fubcomplex induced by the annular vertices, which is canonically identified with the curve complex $C(S)$. It follows from well-known results that this homomorphism is injective. We shall see that for any surface $S_{g,n}$ satisfying $b \leq 1$, this homomorphism is also surjective. In the other cases, we shall describe exactly the structure of $\mathrm{Aut}(D(S)$, and describe the image of the group $\Gamma^*(S)$ in this group.
The complex of domains $D(S)$ is connected except in the cases where S is a sphere with at most four holes or a torus with at most one hole. Indeed, in the non-excluded cases, each vertex of $D(S)$ can be connected by an edge to an annular vertex, and the connectedness of $D(S)$ follows from the connectedness of the curve complex. Furthermore, in all the cases where it is connected, $D(S)$, like $C(S)$ is locally infinite.
Let us analyze two simple examples.

Example 13.2 (The torus with one hole)**.** Let $S = S_{1,1}$, the torus with one hole. Then, $D(S)$ is isomorphic to the product $C(S) \times I$, where $C(S)$ is the curve complex of S and where I is a fixed abstract edge. Indeed, the curve complex $C(S)$ is a discrete set, it is embedded in $D(S)$ as the subcomplex induced by the annular vertices, and, in the case of the torus with one hole, to each vertex of $D(S)$ represented by annulus, one can associate in a canonical way a nonannular vertex, represented by a pair of pants contained in the complement of that annulus.
Thus, $D(S)$ is a fiber space over $C(S)$, with fiber an edge. Note that there are two natural sections of this fiber space, which correspond to taking in each fiber the vertex corresponding to the annular vertex and, respectively, the nonannular vertex.
Thus, for $S = S_{1,1}$, $\mathrm{Aut}(D(S))$ is an extension of the permutation group of the infinite countable set $C(S)$ by the symmetry group of the triangle Δ. Thus, $\mathrm{Aut}(D(S))$ is uncountable, since it contains the permutation group on the infinite set $C(S)$. This implies in particular that $\mathrm{Aut}(D(S))$ is not isomorphic to the extended mapping class group $\Gamma^*(S)$ of S, which is the group $\mathrm{GL}(2, \mathbb{Z})$ and, therefore, is countable.

In fact, if we consider an element of $\mathrm{Aut}(D(S))$ that is supported on one connected component $\{x\} \times I$ of $D(S) \simeq C(S) \times I$, then the symmetry of the edge I which exchanges the two vertices is nongeometric, that is, it is not induced by an extended mapping class.

Example 13.3 (The sphere with four holes)**.** Let $S = S_{0,4}$, the sphere with four holes. Then, $D(S)$ has also a product structure $D(S) \simeq C(S) \times \Delta$, where $C(S)$ is the curve complex of S and where Δ is a fixed abstract 2-dimensional simplex (a triangle). Thus, in this case, $D(S)$ is a fiber space over $C(S)$, with fiber Δ. There is a natural section of this fiber space, which corresponds to taking in each fiber the vertex corresponding to the biperipheral annulus. However, there is no natural way of choosing a biperipheral pair of pants in each fibre. Thus, this fiber space is trivializable but with no natural trivialization.
As in the preceding example, the automorphism group $\mathrm{Aut}(D(S_{0,4}))$ is uncountable, because it contains the permutation group on the infinite set $C(S)$, which implies in particular that $\mathrm{Aut}(D(S_{0,4}))$ is not isomorphic to the extended mapping class group $\Gamma^*(S_{0,4})$ of the sphere with four holes $S_{0,4}$, which is a finite extension of $\mathrm{SL}(2,\mathbb{Z})$ and, therefore, is countable.(The fact that $\Gamma^*(S_{0,4})$ is a finite extension of $\mathrm{SL}(2,\mathbb{Z})$ is proved by considering the sphere as a quotient of the torus by an order-two involution that fixes four points.)
Thus, for $S = S_{0,4}$, $\mathrm{Aut}(D(S))$ is an extension of the permutation group of the infinite countable set $C(S)$ by the symmetry group of the triangle Δ. Note that the elements of $\mathrm{Aut}(D(S))$ that are supported on one connected component $\{x\}\times\Delta$ of $D(S) \simeq C(S)\times\Delta$ form a subgroup which is isomorphic to the symmetry group of the traingle Δ. If such a symmetry exchanges the two nonannular vertices, then it is geometric (that is, it is induced by a mapping class), whereas a symmetry which exchanges an annular and a nonannular vertex is not geometric.

Note the following two features of the complex of domains $D(S)$
1) Unlike $C(S)$, the maximal dimension of a simplex containing a given vertex depends on that vertex (in fact, it depends on the topology of a domain on S representing that vertex).
2) $D(S)$ contains several interesting subcomplexes, for instance, the following:

(1) the complex $C(S)$ (and its subcomplexes);
(2) the complex of pair of pants (that is, the complex induced by the set of vertices that represent domains which are homormorphic to pairs of pants; note that this is not the pants complex that we discussed in Section 7);
(3) the subcomplex induced by the set of nonannular vertices of $D(S)$.

Note that any domain X in S has its own complex of domains $D(X)$ (which may be empty), and that the complex $D(X)$ is a subcomplex of the complex $D(S)$. Any two vertices in $D(X)$ are connected by an edgepath of length

≤ 2 in the complex $D(S)$. (Remember that $X \neq S$, by the definition of a domain.)
Some of these complexes and inclusions are well understood, and the others are certainly worth studying.

Definition 13.4 (The truncated complex of domains)**.** The *truncated complex of domains* $D^2(S)$ *is the subcomplex of* $D(S)$ induced by the set of vertices that are represented by domains which are not biperipheral pairs of pants.

Remarks 13.5. 1) $D^2(S) = D(S)$ if S has at most one boundary component.
2) If S has genus ≥ 1, then $D(S)$ and $D^2(S)$ have the same dimension.
3) Si S has genus 0, then $\dim(D^2(S))) = \dim(D(s)) - 2$.

We first state a result concerning the automorphism group of $D^2(S)$.

Theorem 13.6 (Automorphisms of $D^2(S)$)**.** *Suppose that S is not a sphere with at ≤ 4 holes, a torus with ≤ 2 holes or a closed surface of genus 2. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(D^2(S))$$

is an isomorphism.

Note that since $D^2(S) = D(S)$ if $b \leq 1$, then, Theorem 13.6 is also a theorem about the automorphisms of the complex of domains for surfaces with at most one boundary components (in particular, for closed surfaces). The elaborate part of the proof of Theorem 13.6 is the proof of the surjectivity. I will give here an idea of this proof. It involves the following three steps.

Step 1.— Let ϕ be a simplicial automorphism of $D^2(S)$ and suppose that S is not a torus with one hole. Then, ϕ restricts to a simplicial automorphism of the subcomplex of $D^2(S)$ induced by the set of annular vertices.
The proof of this fact is based on a simplicial characterization of annular vertices, which is given by the following:

Proposition 13.7 (Proposition 6.1 of [49])**.** *Suppose that S is not a torus with one hole and let x be a vertex of $D^2(S)$. Then the following are equivalent:*

(1) x is an annular vertex;
(2) for each vertex y of $D^2(S)$ which is not equal to x, $\mathrm{St}(x, D^2(S))$ is not contained in $\mathrm{St}(y, D^2(S))$.

Step 2.— The restriction of ϕ to the subcomplex of $D^2(S)$ which is induced by the set of annular vertices is induced by an element η of the extended mapping class group of S.
This follows from the Theorem of Ivanov-Korkmaz-Luo on the automorphisms of the curve complex (Theorem 6.3 above), and it uses the fact that

S is neither a sphere with at most four holes nor a torus with at most two holes.

Step 3.— Let $h : S \to S$ be a homeomorphism provided by Step 2, inducing the automorphism η of $C(S)$. We must prove that h induces the automorphism ϕ on $D^2(S)$. To see this, we consider the automorphism $h_* : C(S) \to C(S)$ induced by the homomorphism h of S, and we let $\psi = h_*^{-1} \circ \phi : D^2(S) \to D^2(S)$. We identify the complex $C(S)$ with its image in $D^2(S)$. The automorphism ψ fixes every vertex of $C(S) \subset D^2(S)$, and we must show that ψ is equal to the identity map of $D^2(S)$. This is done by introducing the following new notion:

Definition 13.8. Let x be a vertex of $D(S)$. The *annular link* $\mathrm{Ann}(x)$ *of x in $D(S)$* is the subcomplex of $D(S)$ consisting of those simplices of $\mathrm{Lk}(x, D(S))$ all of whose vertices are annular.

The following proposition is a particular case of Proposition 7.6 of [49].

Proposition 13.9. *Suppose that S is neither a sphere with four holes nor a torus with at most two holes nor a closed surface of genus two, and let x and y be vertices of $D(S)$. Then,* $\mathrm{Ann}(x) = \mathrm{Ann}(y) if and only if x = y$.

Now let v be a vertex of $D^2(S)$. Since ψ is an automorphism of $D^2(S)$ preserving $C(S)$, $\psi(\mathrm{Ann}(v)) = \mathrm{Ann}(\psi(v))$. On the other hand, since $\mathrm{Ann}(v)$ is a subcomplex of $C(S)$ and since ψ fixes each vertex of $C(S)$, $\psi(\mathrm{Ann}(v)) = \mathrm{Ann}(v)$. Hence, $\mathrm{Ann}(\psi(v)) = \mathrm{Ann}(v)$. Since S is not a sphere with four holes, a torus with at most two holes, or a closed surface of genus two, it follows from Proposition 13.9 that $\psi(v) = v$. This proves that ψ is the identity automorphism of $D^2(S)$, which implies that $\phi = h_*$. Hence, the natural homomorphism $\Gamma^*(S) \to D^2(S)$ is surjective.

Next, we study the automorphisms of $D(S)$. For this, we have to deal with the existence of biperipheral pairs of pants.
A *biperipheral edge* of $D(S)$ is an edge whose vertices are a vertex represented by a biperiperal pair of pants and an annulus which is isotopic to the regular neighborhood of the essential boundary component of that pair of pants.
Any automorphism of $D(S)$ induces an automorphism of $D^2(S)$. This is based on the following proposition:

Proposition 13.10. *Suppose that S is not a sphere with four holes, let $\{x, y\}$ be a pair of distinct vertices of $D(S)$ and let $\phi \in \mathrm{Aut}(D(S))$. Then, $\{x, y\}$ is a biperipheral edge if and only if $\{\phi(x), \phi(y)\}$ is a biperipheral edge.*

Note that the automorphism ϕ can exchange the nature of the vertices x and y (it could send the vertex represented by a biperipheral pair of pants to a vertex represented by a biperipheral curve). We shall discuss this fact below.
The proof of Proposition 13.10 is based on the following simplicial characterization of biperipheral edges, which is a particular case of Proposition 9.3 of [49].

Proposition 13.11 (Vertices with the same star in $D(S)$)**.** *Suppose that S is not a sphere with four holes, a torus with at most two holes or a closed surface of genus two, and let x and y be distinct vertices of $D(S)$. Then the following are equivalent:*

(1) $\mathrm{St}(x, D(S)) = \mathrm{St}(y, D(S))$*;*
(2) the pair $\{x, y\}$ is a biperipheral edge of $D(S)$.

To formulate the result on the automorphism group of $D(S)$, we use the following notion that was introduced in [49]:

Definition 13.12 (Simple exchange automorphism)**.** Let K be a simplicial complex, $\{x, y\}$ be a pair of vertices of K, and let $\varphi : K \to K$ be an automorphism of K. We say that φ *is a simple exchange of the complex K, exchanging the vertices x and y*, if $\varphi(x) = y$, $\varphi(y) = x$, and $\varphi(z) = z$ for every vertex z of K which is distinct from x and y.

A pair of vertices x and y satisfying the properties in Definition 13.12 will be called an *exchangeable pair*.
The following follows easily from Proposition 13.11:

Proposition 13.13. *Let K be a simplicial complex. Let $\mathcal{E}$ be a collection of exchangeable pairs of distinct vertices of K with the property that no two distinct pairs in $\mathcal{E}$ have a common vertex. Then there exists a unique automorphism $\varphi_{\mathcal{E}} : K \to K$ such that*

(1) for each pair $\{x, y\}$ in $\mathcal{E}$, $\varphi_{\mathcal{E}}(x) = y$ and $\varphi_{\mathcal{E}}(y) = x$;
(2) $\varphi_{\mathcal{E}}(z) = z$ for every vertex z which is not an element of some pair in $\mathcal{E}$.

Definition 13.14. Let K, $\mathcal{E}$ and $\varphi_{\mathcal{E}} : K \to K$ be as in Proposition 13.13. We call the automorphism $\varphi_{\mathcal{E}} : K \to K$ of K the *generalized exchange of K associated to $\mathcal{E}$*.

Proposition 13.15. *Let K be a simplicial complex and let x and y be vertices of K which are connected by an edge. Suppose that K is a flag complex. Then the following are equivalent:*

(1) x and y are exchangeable in K;
(2) $\mathrm{St}(x, K) = \mathrm{St}(y, K)$.

Proposition 13.16. *Suppose that S is not a sphere with four holes. Then, any two biperipheral edges of S are either equal or disjoint.*

This proposition allows us to define an exchange automorphism associated to an arbitrary set of biperipheral edges.
Let $\mathcal{E}$ be the set of biperipheral edges of $D(S)$. Note that $\mathcal{E}$ is an infinite set unless it is the empty set.
The following is a consequence of Proposition 13.16.

Proposition 13.17. *Suppose that S is not a sphere with four holes. Then, there exists a monomorphism Φ from the set of all subsets of $\mathcal{E}$ into* $\mathrm{Aut}(D(E))$

such that for each subset $\mathcal{F}$ of biperipheral edges of $D(S)$, $\Phi(\mathcal{F})$ is the automorphism which exchanges the two vertices of each biperipheral edge of $\mathcal{F}$ and fixes each vertex of $D(S)$ that is not the vertex of a biperipheral edge in $\mathcal{F}$.

Using this proposition and using Theorem 13.6, we obtain the following:

Theorem 13.18. *Suppose that S is not a sphere with at most four holes, a torus with at most two holes, or a closed surface of genus two. and let $\mathcal{E}$ be the collection of biperipheral edges of $D(S)$. Let $\varphi \in \mathrm{Aut}(D(S))$. Then there exists a unique subset $\mathcal{F}$ of $\mathcal{E}$ and a unique element γ of $\Gamma^*(S)$ such that $\varphi = \varphi_{\mathcal{F}} \circ \gamma_*$ where $\varphi_{\mathcal{F}}$ is the exchange automorphism of $D(S)$ corresponding to $\mathcal{F}$ and γ_* is the geometric automorphism of $D(S)$ induced by γ.*

Part 3. ACTIONS ON SPACES OF FOLIATIONS

14. Measured foliations

This section contains a short review of the theory of measured foliations that is necessary to understand the rigidity results that we present in the following sections.

The basic theory is explained in [16].

For simplicity of the exposition, and to avoid talking about singular points on the boundary, we shall limit ourselves to the case where S is a closed surface of genus $g \geq 1$, but the theory works as well for surfaces with boundary.

We consider foliations with singularities on S, where the singular points are "generalized saddles", that is, isolated points with k separatrices where all values of $k \geq 3$ are allowed. The local models at the singular points are given in Figure 10 below.[4]

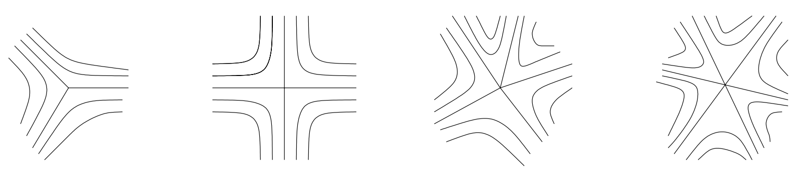

Figure 10. The four pictures represent singular points of measured foliations, with k-separatrices, for $k = 3, 4, 5, 6$ respectively.

Note that provided the surface S is not the torus, any foliation on S has at least one singular point.

A *transverse measure* for a foliation is a measure on each transverse arc that is equivalent to the Lebesgue measure of an interval of $\mathbb{R}$, such that the measure on arcs is invariant by the local holonomy maps, that is, by isotopies of arcs that keep each point on the same leaf.

[4] Bill Browder suggests that, because of the local model at the singular points, we call such foliations *graph foliations*.

A *Whitehead move* is an operation on measured foliations that consists in contracting to a point a compact leaf that joins two singular points, or the inverse move. An example of a Whitehead move is represented in Figure 11. The equivalence relation between measured foliations that is generated by isotopy and Whitehead moves is called *Whitehead equivalence*.
The space of Whitehead equivalence classes of measured foliations is called *measured foliations space*, and it is denoted by $\mathcal{MF}$ or $\mathcal{MF}(S)$.
Given a measured foliations F on S, we shall use the notation $[F]$ for its equivalence class in $\mathcal{MF}$.

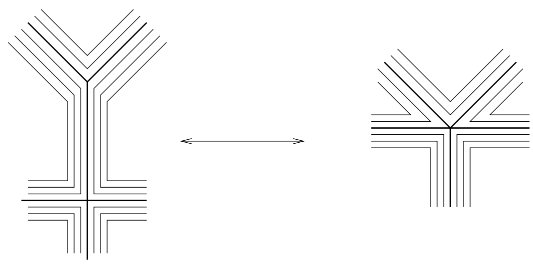

FIGURE 11. Whitehead move: collapsing or creating an arc joining two singular points.

It will also be convenient to represent elements of $\mathcal{MF}$ by *partial* measured foliations, that is, measured foliations whose supports are nonempty (and not necessarily connected) subsurfaces with boundary of S. We shall denote by $\mathrm{Supp}(F)$ the support of a partial measured foliation F. If F is a partial measured foliation, then, a genuine measured foliations F_0 (which we sometimes call a *total* measured foliation, to stress the fact that its support is equal to S) is obtained from F by collapsing each connected component of $S \setminus \mathrm{Supp}(F)$ onto a spine. The equivalence class of F_0 does not depend on the choice of the spines (because different spines of a surface with boundary differ by a Whitehead move). In this way, a partial measured foliation gives a well defined element of $\mathcal{MF}$.
We let $\mathcal{S}$ be the set of isotopy classes of essential curves on S.
THis is a natural map from $\mathbb{R}_+^* \times \mathcal{S}$ into $\mathcal{MF}$, which is defined as follows. Given a pair $(r, [c])$ where $r > 0$ and $[c]$ is the isotopy class of an essential curve c on S, we consider an annulus N in the interior of S which is isotopic to a regular neighborhood of c, and we foliate N by closed leaves which are homotopic to c. We obtain a foliated annulus, and we equip it with a transverse measure such that the total transverse measure of a segment that joins the two boundary components of that annulus and that is tranverse to the foliation is equal to r. The result is a partial measured foliation on S, which, by the above discussion, gives a well defined element of $\mathcal{MF}$. The resulting map $\mathbb{R}_+^* \times \mathcal{S} \to \mathcal{MF}$ is injective.
We also need to talk about the components of a measured foliation.

Every equivalence class of measured foliations has a well defined finite number of components, defined as follows.
Let F be a measured foliation on S. The *singular graph* K of F is the union of the compact leaves that join the singular points of F. We define the *components* of F as the (partial) measured foliations that are the closures of the connected components of $S \setminus K$. In this way, each measured foliation on S can be decomposed into a union of finitely many components. Each component is a partial measured foliation which provides a well defined element of $\mathcal{MF}$. The set of equivalence classes of the components of F depends only on the equivalence class of F.
We now recall the definition of the geometric intersection function $\mathcal{MF} \times \mathcal{S} \to \mathbb{R}_+$.
For any measured foliation F and for any element γ of $\mathcal{S}$, we define $i(F,\gamma)$ to be the infimum of the total measure of c over all closed curves c in the homotopy class γ which are made up of arcs transverse to F and arcs in the leaves of F. (To compute the total measure of c, we consider the transverse measure of the arcs of c that are contained in the leaves of F to be equal to 0.)
The quantity $i(F,\gamma)$ does not depend on the choice of the Whitehead equivalence class of F. Thus, we obtain a map $i(.,.)$ defined on the product $\mathcal{MF} \times \mathcal{S}$, called the *geometric intersection function.*
Using this intersection function, we naturally obtain a natural map from the space $\mathcal{MF}$ to the space $\mathbb{R}_+^{\mathcal{S}}$ of nonnegative functions on $\mathcal{S}$. This map is injective. In other words, the collection of intersection functions of a measured foliation class with the set of all isotopy classes of simple closed curves uniquely determines the measured foliation class.
The set $\mathcal{MF}$ inherits a topology from this injection into $\mathbb{R}_+^{\mathcal{S}}$, which makes it homeomorphic to $\mathbb{R}^{6g-6} \setminus \{0\}$. It is sometimes useful to consider the "empty foliation" as an element of $\mathcal{MF}$ (and we shall do this in the next section). In this case, the space $\mathcal{MF}$ is homeomorphic to $\mathbb{R}^{6g-6}$.
There is a natural action of the set of positive reals $\mathbb{R}_+^*$ on $\mathcal{MF}$, obtained from this action of $\mathbb{R}_+^*$ on measured foliations defined by multiplying the transverse measure by a constant positive factor.
The quotient space of $\mathcal{MF}$ by the action of $\mathbb{R}_+^*$ is denoted by $\mathcal{PMF}$.
From the embedding of $\mathcal{MF}$ into the space $\mathbb{R}_+^{\mathcal{S}}$ we obtain an embedding of $\mathcal{PMF}$ in the projectived space $P\mathbb{R}_+^{\mathcal{S}}$.
Finally, we recall that the intersection function $i : \mathcal{MF} \times \mathcal{S} \to \mathbb{R}_+$ extends continuously to an intersection function defined on $\mathcal{MF} \times \mathcal{MF}$ which is denoted by the same letter:

$$i : \mathcal{MF} \times \mathcal{MF} \to \mathbb{R}_+.$$

15. Automorphisms that preserve intersection functions

For each α in $\mathcal{F}$, let $i_\alpha : \mathcal{MF} \to \mathbb{R}_+$ be the associated intersection function. The following definitions are made by F. Luo in [40].

Definition 15.1 ($\mathcal{F}$-structure)**.** An $\mathcal{F}$-structure on a topological space X is a collection $\mathcal{F}$ of (real or complex-valued) functions defined on X, such that the collection

$$\{f^{-1}(U) \mid U \text{ is open in } \mathbb{R} \text{ (respectively in } \mathbb{C}) \text{ and } f \in \mathcal{F}\}$$

of subsets of X is a sub-basis for the topology of that space.

Definition 15.2. Given an $\mathcal{F}$-structure on a toplogical space X, an *automorphism* of this structure is a homeomorphism $\phi : X \to X$ satisfying $\phi^*(\mathcal{F}) = \mathcal{F}$ where $\phi^*(\mathcal{F}) = \{f \circ \phi \mid f \in \mathcal{F}\}$.

Luo proves the following:

Theorem 15.3 (Luo [40])**.** *Let S be a closed surface S of genus ≥ 2 and consider the $\mathcal{F}$-structure*

$$\mathcal{F} = \{i_\alpha \mid \alpha \in \mathcal{S}\}$$

on $\mathcal{MF}$, where $i_\alpha : \mathcal{MF} \to \mathbb{R}_+$ denotes the intersection function. Then, any automorphism of $(\mathcal{MF}, \mathcal{F})$ is induced by an element of the extended mapping class group.

As a corollary, Luo obtains the following:

Corollary 15.4 (Luo [40])**.** *Let S be a closed surface of genus ≥ 2. For each $\alpha \in \mathcal{S}$, let PZ_α be the image in $\mathcal{PMF}$ of the set $Z(\alpha) = \{F \in \mathcal{MF} \mid i(\alpha, F) \neq 0\}$ by the natural quotient map $\mathcal{MF} \to \mathcal{PMF}$. Then, any homeomorphism of $\mathcal{PMF}$ preserving the collection $\{PZ_\alpha \mid \alpha \in \mathcal{S}\}$ of subsets of $\mathcal{PMF}$ is induced by an element of the extended mapping class group.*

We note that Luo's results in [40] also include the cases of surfaces with boundary.
The proof by Luo of these results uses the theorem of Ivanov, Korkmaz and Luo on the automorphism of the complex of curves (Theorem 6.3).
In the same paper, Luo considers the $\mathcal{F}$-structure on Teichmüller space consisting of length functions, and he states a result which is analogous to Theorem 15.4, that is, any automorphism of Teichmüller space that preserves the $\mathcal{F}$-structure of length functions is induced by an element of the extended mapping class group.
Finally, we note that Luo also defined on the space of conjugacy classes of representations of the fundamental group of a closed surface in $\mathrm{SL}(2, \mathbb{C})$ an $\mathcal{F}$-structure associated to the trace functions $\{tr_\alpha \mid \alpha \in \mathcal{S}\}$, and he asked the question of stydying the automorphism group of that structure.

16. Train tracks

We shall review the notion of train track and we shall recall some basic facts about the train track piecewise linear (PL) structure of the space $\mathcal{MF}$ of equivalence classes of measured foliations on a surface. For simplicity, we shall suppose that $S = S_g$ is a closed oriented surface of genus $g \geq 2$.

The notion of train track was introduced by Thurston in [67]. A standard reference on train tracks is the monograph [61] by Penner and Harer.
A *train track* τ on S is a graph embedded in S whose vertices are all trivalent and such that the three half-edges that abut on any vertex have a well defined tangent at that point. The local structure at a vertex is represented in Figure 12. (In fact, no smooth structure on S is really needed to define a train track, and instead of talking about a tangent direction at a vertex, one can simply say that there is a well defined notion of "two half-edges abutting from the same side" and "one half-edge abutting from the other side" at each vertex.)
We note that a train track can have no vertices, i.e. it could consist of a union of disjoint simple closed curves, and in that case we shall also call such a closed curve an edge of the train track.
A vertex of τ is also called a *switch*. We shall call a *corner* of S a region in a neighborhood of a switch which is contained between the two half-edges that abut from the same side. All the train tracks τ that we consider satisfy the following property: any connected component of $S \setminus \tau$ is not a disk with 0, 1 or 2 corners or an annulus with no corner (cf. Figure 13).

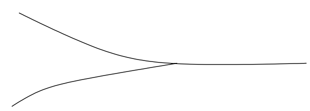

FIGURE 12. The local model at a switch.

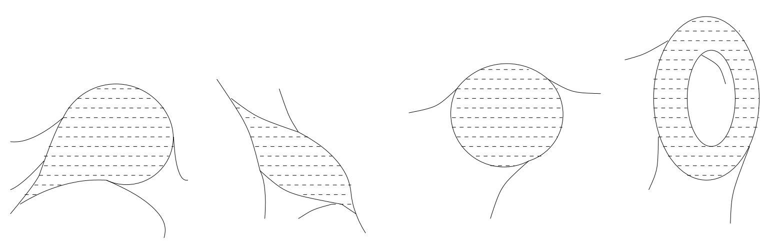

FIGURE 13. The shaded regions represent the four types of excluded components of the complement of a train track.

A train track τ is said to be *maximal* if every component of $S \setminus \tau$ is a disk with three corners on its boundary.
Any train track τ has a *regular neighborhood* $N(\tau)$ foliated by arcs that are called the *ties*. The local picture of the foliation by the ties near a switch is represented in Figure 14. The regular neighborhood $N(\tau)$, equipped with its foliation by the ties, is well defined up to isotopy, and there is a natural projection $N(\tau) \searrow \tau$ which is defined by collapsing every tie to a point.
Let τ be a train track and let $e_1, \ldots, e_N$ be its edges. Let $\mathbb{R}^N$ denote the real vector space with basis $\{e_1, \ldots, e_N\}$ and let $(x_1, \ldots, x_N)$ denote the

coordinates of a point in that space. We let $V_\tau \subset \mathbb{R}^N$ denote the closed convex cone in $\mathbb{R}^N$ defined by the system

$$\begin{cases} x_i \geq 0 & \text{for every } i = 1, \ldots, n \\ x_i = x_j + k_k & \text{for every switch of } \tau \end{cases}$$

where, in the equations, x_j and x_k denote the weights at the two edges that abut from the same side at the given switch, and x_i is the weight on the edge that abuts from the other side on that switch.
A train track τ is said to be *recurrent* if there exists an element $(x_1, \ldots, x_N)$ of V_τ satisfying $x_i > 0$ for all $i = 1, \ldots, N$.
Let τ be a train track on S. There is a map

$$\varphi_\tau : V_\tau \to \mathcal{MF}$$

defined as follows. Let $N(\tau)$ be a regular neighborhood of τ equipped with its projection $N(\tau) \searrow \tau$. Let $(x_1, \ldots, x_N)$ be a nonzero element of V_τ. For each nonzero coordinate x_i, consider the inverse image of the edge e_i by the projection $N(\tau) \searrow \tau$. The closure of the interior of this inverse image has a natural structure of a rectangle equipped with a foliation induced by the ties, which we call the "vertical" foliation. We equip this rectangle with another foliation, which we shall call the "horizontal" foliation, whose leaves are segments which are transverse to the ties and which join the two edges of the rectangle that consist of ties. We equip this horizontal foliation with a transverse measure whose total mass is equal to x_i. We can glue the various foliated rectangles along their vertical sides using measure-preserving homeomorphisms. We obtain a partial measured foliation on S, which, by the construction described in Section 14, gives a well defined element of $\mathcal{MF}$. The zero element of V_τ is sent to the empty foliation of $\mathcal{MF}$. This defines the map $\varphi_\tau : V_\tau \to \mathcal{MF}$. This map is a homeomorphism onto its image, and in the case where τ is maximal and recurrent, the image $\varphi_\tau(V_\tau) = U_\tau$ has nonempty interior in $\mathcal{MF}$ (see [55] p. 20).
A measured foliation F (or its equivalence class $[F] \in \mathcal{MF}$) is said to be *carried by* a train track τ if $[F]$ is in the image U_τ of V_τ by the map φ_τ.

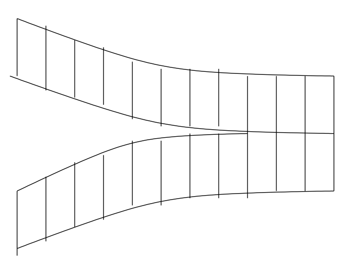

FIGURE 14. The regular neighborhood and the local structure of the ties near a switch

If τ and σ are two train tracks on S, we say that τ is *carried* by σ, and we write this relation as $\tau \prec \sigma$, if τ is isotopic to a train track τ' which is contained in a regular neighborhood $N(\sigma)$ of σ and which is transverse to the ties. When $\tau \prec \sigma$, there is a natural linear map from the closed convex cone V_τ to the closed convex cone V_σ which induces the inclusion map at the level of the two subspaces $\varphi_\tau(V_\tau)$ and $\varphi_\sigma(V_\sigma)$ of $\mathcal{MF}$.

17. Automorphisms of the train track PL structure

We shall consider Thurston's piecewise-linear structure of the space $\mathcal{MF}$. This structure is defined by an atlas. We note by the way that although a PL structure is rigid in some sensenot a (G, X) structure in the usual sense. Indeed, there is no such thing as a "universal PL manfiold" X in each dimension equipped with an automorphism group G with the property that any local isometry of X extends in a unique manner to an element of G.

Let M be a nonnegative integer. We define a *linear polytope* V in $\mathbb{R}^M$ to be a subset of $\mathbb{R}^M$ which is equal to the intersection of a finite number of closed linear half-spaces. Let us stress on the fact that we are talking about linear and not just affine half-spaces, so that V is $\mathbb{R}$-homogeneous (invariant by multiplication by elements of $\mathbb{R}$). In particular, it is noncompact (unless it is empty).

The dimension of a linear polytope V is the smallest dimension of a vector space in which V embeds linearly.

We shall use the name *polytope* to denote a linear polytope.

Note that a linear polytope is closed and convex.

The *relative interior* of a convex set V in $\mathbb{R}^n$ is the topological interior of $V \cap A$ in A, with A being the smallest (with respect to inclusion) affine subset of $\mathbb{R}^n$ containing V.

The dimension of a linear polytope is the dimension of the smallest affine subset of $\mathbb{R}^n$ that contains it.

Definition 17.1 (PL function)**.** Let M and N be two nonnegative integers, let V be a finite union of polytopes $V_1, \ldots, V_n$ in $\mathbb{R}^M$ having the same dimension. A function $f : V \to \mathbb{R}^N$ is said to be piecewise-linear (PL for brevity) if f is continuous and if the restriction of f to the relative interior of each of the polytopes $V_1, \ldots, V_n$ is the restriction of a linear function from $\mathbb{R}^M$ to $\mathbb{R}^N$.

Note that in Definition 17.1, the union V of the polytopes $V_1, \ldots, V_n$ is equipped with the topology induced from that of $\mathbb{R}^M$. If $V_1, \ldots, V_n$ are disjoint, then the continuity of f boils down to the continuity of its restriction to each set V_i, whereas if $V_1, \ldots, V_n$ are not disjoint, a further condition is required at the intersection points.

We now consider the PL structure on $\mathcal{MF}$ defined by the train track coordinates.

For each maximal recurrent train track τ, we let $\psi_\tau : U_\tau \to V_\tau$ denote the inverse of the homeomorphism $\varphi_\tau : V_\tau \to U_\tau$.
Let

$$\mathcal{A} = \{(U_\tau, \psi_\tau) \mid \tau \text{ is a maximal recurrent train track } \}.$$

Theorem 17.2 (Thurston)**.** *The set $\mathcal{A}$ is an atlas of a PL structrure on $\mathcal{MF}$.*

We shall define precisely what we mean by an automorphism of this train track PL structure.
We shall need a precise description of the coordinate changes $\psi_{\tau\sigma} = \psi_\sigma \circ \psi_\tau^{-1}$ (defined on the appropriate subset of V_τ) of the atlas $\mathcal{A}$. This description is given in Chapter 1 of [55] and we shall recall it here. For that, we introduce the following notion.
Let τ be a maximal recurrent train track on S and let $T = \{\tau_1, \ldots, \tau_n\}$ be a family of maximal recurrent train tracks. We say that the family T is *adapted to* τ if the following properties are satisfied:

(1) For each $i = 1, \ldots, k$, the train track τ_i is maximal and recurrent.
(2) For each $i = 1, \ldots, k$, $\tau_i \prec \tau$.
(3) For every i and j satisfying $1 \leq i < j \leq k$, the interiors of U_{τ_i} and U_{τ_j} are disjoint. (Equivalently, for every such i and j, the images in U_τ of the interiors of the convex cones $V_{\tau_i} \subset V_\tau$ by the natural maps induced by the relations $\tau_i \prec \tau$) (2) are disjoint.)

Proposition 17.3. *Let (U_τ, ψ_τ) and (U_σ, ψ_σ) be two coordinate charts in $\mathcal{A}$ and let $\psi_{\tau,\sigma}$ be the corresponding coordinate change, defined on the subset $\psi_\tau(U_\tau \cap U_\sigma)$ of V_τ.*
For every point F in the interior of $\psi_\tau(U_\tau \cap U_\sigma)$, we can find a family $T = \{\tau_1, \ldots, \tau_n\}$ of train tracks which is adapted to τ and which furthermore satisfies the following properties:

(1) $\tau_i \prec \sigma$ for all $i = 1, \ldots, k$.
(2) The union $\bigcup_{i=1}^k U_{\tau_i}$ is a neighborhood of F is $\mathcal{MF}$, and F belongs to each set U_{τ_i}, for all $i = 1, \ldots, k$.
(3) The linear maps that define the PL map $\psi_{\tau\sigma}$ are constituted by the linear maps induced by the relations $\tau_i \prec \tau$ and $\tau_i \prec \sigma$, and the inverses of such maps.

We note that Property (3) of Proposition 17.3 implies that the restriction of the coordinate change of coordinates $\psi_{\tau,\sigma}$ to the neighborhood $N(F)$ of F is linear on each subset U_{τ_i} of $N(F)$. Proposition 17.3 is the basic technical result which implies that the coordinate changes in the atlas $\mathcal{A}$ are piecewise-linear.
The study of the automorphisms of the train track PL structure will be based on considerations on the singular set of a PL function.

Definition 17.4 (Singular set of a PL function)**.** We use the notations of Defintion 17.1. Let $f : V \to \mathbb{R}^N$ be a PL function. The singular set of f, denoted by $\mathrm{Sing}(f)$, is the set of points $x \in V_1 \cup \ldots \cup V_n$ such that f is not linear in any neighborhood of x.

We note the following two observations:
1) It follows from Definition 17.1 that the set $\mathrm{Sing}(f)$ is a union of codimension one faces which are intersections of two sets in the collection of polytoes $\{V_1, \ldots, V_n\}$.
2) If the polytopes V have dimension D, then $\mathrm{Sing}(f)$ has a natural structure of a union of linear polytopes of dimension $D-1$ in $\mathbb{R}^M$, and the restriction of f to $\mathrm{Sing}(f)$ is PL.
Thus, there exists a nested sequence of subsets of V,

$$\mathrm{Sing}_0(f) \supset \mathrm{Sing}_1(f) \supset \ldots \mathrm{Sing}_k(f),$$

where, by definition,

(N1) $\mathrm{Sing}_0(f) = \mathrm{Sing}_0(f) = V \setminus \mathrm{Sing}(f)$;
(N2) $\mathrm{Sing}_1(f) = \mathrm{Sing}(f)$;
(N3) for every integer i satisfying $2 \leq i \leq k$, $\mathrm{Sing}_i(f)$ is the singular set of the restriction of f to $\mathrm{Sing}_{i-1}(f)$;
(N4) the restriction of f to each component of $\mathrm{Sing}_k(f)$ is linear.

We note that for each $2 \leq i \leq k$, $\mathrm{Sing}_i(f)$ is a codimension-1 subset of $\mathrm{Sing}_{i-1}(f)$ (see Observation 2 above) and that $k \leq n$.
Let f be a PL function defined on a set V as above. The associated sequence $\mathrm{Sing}_0(f) \supset \mathrm{Sing}_1(f) \supset \ldots \mathrm{Sing}_k(f)$ defines a stratification of V, each stratum having a well defined codimension in V.
We shall call this stratification the *flag* induced by f on the set V, and we shall denote this structure by $\mathrm{Fl}(f)$.

Definition 17.5 (Train track PL function)**.** Let N be a nonnegative integer. A function $f : \mathcal{MF} \to \mathbb{R}^N$ is said to be a *train track* PL *function* if for every x in $\mathcal{MF}$, there exists a chart (U_τ, ψ_τ) belonging to the atlas $\mathcal{A}$ such that the set U_τ contains x in its interior, such that the function $f \circ \psi_\tau^{-1}$ defined on $V_\tau = \psi_\tau(U_\tau)$ is PL, and such that there exists a coordinate change map $\psi_{\tau\sigma}$ belonging to the atlas $\mathcal{A}$, having $\psi(x)$ in the interior of its domain and such that the singular sets of the restrictions of the maps $f \circ \psi_\tau$ and $\psi_{\tau\sigma}$ to the set V_τ coincide in a neighborhood of $\psi_\tau(x)$ in V_τ.

Let $\mathcal{P}$ be the set of train track PL functions on $\mathcal{MF}$.
In some sense, the set $\mathcal{P}$ is the set of smoothest possible functions on $\mathcal{MF}$ relatively to the atlas $\mathcal{A}$.

Definition 17.6 (Automorphism of the PL structure)**.** We shall say that a homoemorphism $h : \mathcal{MF} \to \mathcal{MF}$ preserves the set $\mathcal{P}$ of train track PL functions if for every element f of $\mathcal{P}$, $f \circ h$ is also in $\mathcal{P}$.

We denote by $\mathrm{Aut}(\mathcal{MF}, \mathcal{P})$ the group of homeomorphisms of $\mathcal{MF}$ that preserve the set $\mathcal{P}$.

Proposition 17.7. *The action on $\mathcal{MF}$ of any element of the extended mapping class group preserves the set of train track* PL *functions.*

Proof. Let $h : \mathcal{MF} \to \mathcal{MF}$ be a homeomorphism induced by an extended mapping class, let N be a nonnegative integer and let $f : \mathcal{MF} \to \mathbb{R}^N$ be a function in $\mathcal{P}$. For each x in $\mathcal{MF}$, let (U_τ, ψ_τ) and $\psi_{\tau\sigma}$ be respectively a chart in $\mathcal{A}$ and a coordinate change map in $\mathcal{A}$ satisfying the properties required in Definition 17.5. Then, $\tau' = h(\tau)$ and $\sigma' = h(\sigma)$ are maximal recurrent train tracks on S such that $(h(U_\tau), \psi_{\tau'})$ is a chart in $\mathcal{A}$ and $\psi_{\tau'\sigma'}$ is a coordinate change in $\mathcal{A}$, which also satisfy the properties required in Definition 17.5, in the neighborhood of the point $h(x)$ instead of the point x. This completes the proof. □

From Proposition 17.7, we have a homomorphism

$$\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF}, \mathcal{P}).$$

The main result in the rest of this section (Theorem 17.10 below) says that this homomorphism is an isomorphism, except in genus two, where the homomorphism is surjective with kernel $\mathbb{Z}_2$. Before proving this theorem, we need to establish a few more results.

A *system of curves* on S is the isotopy class of a collection of disjoint and pairwise non-isotopic essential curves on S. Note that the number of elements in such a collection is bounded above by $3g - 3$.

For every integer k satisfying $1 \leq k \leq 3g - 3$, we denote by of $\mathcal{S}_k$ the subset of $\mathcal{S}'$ consisting of isotopy classes of systems of curves that are representable by a collection of curves of cardinality k.

Note that in particular $\mathcal{S}_1 = \mathcal{S}$.

For each k satisfying $1 \leq k \leq 3g-3$, let $\mathcal{MF}_k \subset \mathcal{MF}$ be the set of measured foliation classes x having the following properties:

(C1) For each i satisfying $0 \leq i \leq k - 1$, there does not exist any chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain U_τ such that there exists a coordinate change $\psi_{\tau\psi}$ having $\psi(x)$ in the interior of its domain, such that $\psi(x)$ is on a stratum of dimension i of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$.

(C2) There exists a coordinate chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain and a coordinate change $\psi_{\tau\psi}$ having $\psi_\tau(x)$ in the interior of its domain, such that $\psi(x)$ is on a stratum of dimension k of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$, and such that $\psi_\tau(x)$ is a convex combination of k elements in the 1-stratum of $\mathrm{Fl}(\psi_{\tau\sigma})$ with respect to the linear structure of V_τ induced from its inclusion inclusion in $\mathbb{R}^N$.

Note that, in particular, $\mathcal{MF}_1 \subset \mathcal{MF}$ is simply the set of measured foliation classes x such that there exists a coordinate chart (U_τ, ψ_τ) in $\mathcal{A}$ having x in the interior of its domain and a coordinate change $\psi_{\tau\psi}$ having $\psi(x)$ in the interior of its domain such that x is on a stratum of dimension 1 of the flag defined by the singular set $\mathrm{Fl}(\psi_{\tau\sigma})$.

Proposition 17.8. *For any $k \geq 0$, any automorphism of $(\mathcal{MF}, \mathcal{P})$ preserves the set $\mathcal{MF}_k$.*

Proof. An automorphism of $(\mathcal{MF}, \mathcal{P})$ acts on the set of flags of the coordinate changes $\psi_{\tau\sigma}$ of $\mathcal{A}$, that is, it carries the flag of any coordinate change in $\mathcal{A}$ to a flag of some coordinate change in $\mathcal{A}$, and it preserves the properties defining the elements of $\mathcal{MF}_k$, for each $k \geq 1$. □

For each integer k satisfying $1 \leq k \leq 3g-3$, there is a natural inclusion $j_k : (\mathbb{R}_+^*)^k \times \mathcal{S}_k \hookrightarrow \mathcal{MF}$, defined by associating to each $v \in (\mathbb{R}_+^*)^k$ and to each element $C \in \mathcal{S}_k$ the equivalence class of a partial measured foliation F with the following properties:

(1) the support of F is the union of disjoint annuli $A_1, \ldots, A_k$ which are foliated by closed leaves
(2) each annulus A_i is a regular neighborhood of a closed curve c_i, where $c_1, \ldots, c_k$ are the components of a system of curves representing the isotopy class C;
(3) for $1 \leq i \leq k$, the total transverse measure of the annulus A_i is equal to the i-the coordinate of v.

We shall call a foliation on S representing an element of $\mathcal{MF}$ which is the image of some element of $\mathcal{S}'$ by one of the maps j_k an *annular foliation*.

Proposition 17.9. *For every $k \geq 1$, the image of $(\mathbb{R}^*)^k \times \mathcal{S}_k$ in $\mathcal{MF}$ is the set $\mathcal{MF}_k$.*

Proof. Let $F \in \mathcal{MF}$ be a measured foliation class which is in the image of $(\mathbb{R}^*)^k \times \mathcal{S}_k$. We must show that it satisfies Properties (C1) and (C2) above. The proof uses a technique used in [55] Chapter 1. The idea is as follows. We start by representing F by a system of weights on a train track consisting of a union of k disjoint simple closed curves, representing the element of $(\mathbb{R}^*)^k \times \mathcal{S}_k$ that defines F. We can pinch this system of curves along a system of disjoint arcs having their endpoints on these curves in order to obtain a maximal recurrent train track τ such that F is in the interior of the linear polytope V_τ. A pinching operation is represented in Figure 15. By choosing a different system of arcs, we can obtain a maximal recurrent train track σ such that F is in the interior of the linear polytope V_σ, and we can choose this new system of arcs so that F is in the codimension-k skeleton of the flag $\mathrm{Fl}(\psi_{\tau\sigma})$. This uses the description of the coordinate changes that is contained in Proposition 17.3 above.
Conversely, it is easy to see that if a measured foliation satisfies Properties (C1) and (C2), then it is in the image of $(\mathbb{R}^*)^k \times \mathcal{S}_k$.

□

Theorem 17.10. *Suppose that S is not the closed surface of genus 2. Then, the homomorphism*

$$\Gamma^*(S) \to \mathrm{Aut}(\mathcal{MF}, \mathcal{P})$$

is an isomorphism. In the special case where S is a closed surface of genus 2, this isomorphism is surjective, and its kernel is $\mathbb{Z}_2$ generated by the hyperelliptic involution.

Proof. Let f be an element of $\mathrm{Aut}(\mathcal{MF},\mathcal{P})$. By Proposition 17.8, f preserves the subset $\mathcal{MF}_1$ of $\mathcal{MF}$. Proposition 17.9 says in particular that $\mathcal{MF}_1$ is the natural image of $\mathbb{R}_+^*\times\mathcal{S}$ in $\mathcal{MF}$, that is, it is the set of measured foliation classes that are representable by foliations all whose nonsingular leaves are closed curves homotopic to a single simple closed curve. Thus, $\mathcal{MF}_1$ is in natural one-to-one correspondence with the set $\mathbb{R}_+^*\times\mathcal{S}$ of isotopy classes of weighted essential curves on S. Since f acts linearly on rays, it acts on the set $\mathcal{S}$ of isotopy classes of essential curves, which is the set of verices of the curve complex $C(S)$. Therefore, f defines in a natural manner a map from the vertex set of the curve complex $C(S)$ to itself, and it follows from the fact that f is a homeomorphism that the map defined on $C(S)$ is a bijection.
Similarly, by Proposition 17.8, for each $k=2,\ldots,3g-3$, f preserves the set $\mathcal{MF}_k$ of $\mathcal{MF}$ which, again by Proposition 17.9, can be naturally identified with the set $\mathcal{S}_k$ of isotopy classes of weighted systems of curves which have k components, and f induces also a map of the set of $(k-1)$-simplices of $C(S)$. Thus, the bijection induced by f on the vertex set of $C(S)$ extends naturally to a simplicial automorphism of $C(S)$.
By the result of Ivanov (Theorem 6.2), the action of f on $C(S)$ is induced by an element γ of the extended mapping class group of S, and it is clear from the definitions of these actions that the restriction of f and of the extended mapping class γ on the image of $\{1\}\times\mathcal{S}$ (and, even, of $\{1\}\times\mathcal{S}'$) in $\mathcal{MF}$ coincide. Since f and γ are linear on rays of $\mathcal{MF}$, these actions coincide on the subset $\mathbb{R}_+^*\times\mathcal{S}$. Since the image of $\mathbb{R}_+^*\times\mathcal{S}$ in $\mathcal{MF}$ is dense and since the actions of f and γ on $\mathcal{MF}$ are continuous, they coincide on $\mathcal{MF}$. Thus, each automorphism of $(\mathcal{MF},\mathcal{P})$ is induced from an extended mapping class. This proves the surjectivity of the homomorphism $\Gamma^*(S)\to\mathrm{Aut}(\mathcal{MF},\mathcal{P})$. The statement about the kernel can be deduced from the fact that the homomorphism from the extended mapping class group to the automorphism group of the complex of curves is injective except in genus two, in which case the kernel is $\mathbb{Z}_2$. $\square$

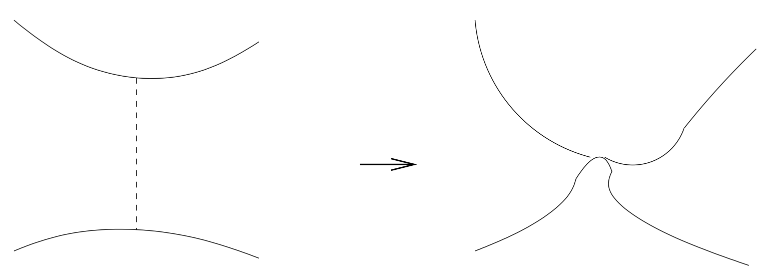

FIGURE 15. The pinching operation that is used in the proof of Proposition 17.9.

18. The symplectic structure of $\mathcal{MF}$

In this section, S is also a closed surface. Let τ be a maximal recurrent train track on S. We use the notations established in Section 16. In particular, $\mathbb{R}^N$ is the vector space with basis the set of edges $\{e_1, \ldots, e_N\}$ of τ.
Let $F(\tau)$ be the vector subspace of $\mathbb{R}^n$ defined by the switch equations

$$x_i = x_j + k_k \text{ for every switch of } \tau,$$

where
where, as above, x_j and x_k denote the weights at the two edges that abut from the same side, and x_i is the weight on the edge that abuts from the other side on that switch.
Note that the cone V_τ defined in Section 16 is the cone of vectors in $F(\tau)$ whose coordinates are all nonnegative. The vector space $F(\tau)$ can be seen at each point as the tangent space to $\mathcal{MF}$ at that point, in the chart associated to the train track τ.
There is a bilinear form $\langle .,. \rangle$ defined on each vector space $F(\tau)$, whose existence has been pointed out by Thurston, and we now recall its definition. The surface S being oriented, we can talk, at each switch of τ, of the half-edge abutting from the left and the one abutting from the right on that switch, with respect to an observer sitting between the two edges abutting from the same side, and looking at the switch. Let a_l and a_r denote respectively these two half-edges.
Let $X = (x_1, \ldots, x_N)$ and $Y = (y_1, \ldots, y_N)$ be two elements in $F(\tau) \subset \mathbb{R}^N$. The product $\langle X, Y \rangle$ is defined as

$$\langle X, Y \rangle = \frac{1}{2} \sum (x_l y_r - x_r y_d)$$

where the sum is over all the switches of τ, and where for each switch, the indices l and r are the weights induced by the corresponding vectors on the edges a_l and a_r respectively, that abut from the same side on that switch.
The form $\langle .,. \rangle$ associated to a maximal recurrent train track is closed and nondegenerate, and it defines a PL symplectic structure on $\mathcal{MF}$, which is invariant by the train track PL coordinate changes, that is, the coordinate changes of the atlas $\mathcal{A}$ defined in Section 16. In particular, this form is invariant by the action of the mapping class group $\Gamma(S)$.
Some of the properties of this form are studied in the papers [56], [57] and [58] and in the monography [61].
There is a natural homomorphism from the mapping class group $\Gamma(S)$ into the symplectomorphism group $\mathrm{Sym}(\mathcal{MF})$ of the space $(\mathcal{MF}, \langle .,. \rangle)$. We ask the question of characterizing the symplectomorphism group $\mathrm{Sym}(\mathcal{MF})$ and of descrfibing the image and the kernel of the natural homomorphism $\Gamma(S) \to \mathrm{Sym}(\mathcal{MF})$.

19. Automorphisms of the space of unmeasured foliations

In this section, as in the preceding two sections, we take the surface S to be closed. In [59] there is a complete treatment of the result on the automorphism group that we prove here in the general case of surfaces with punctures.

Let $\mathcal{UMF} = \mathcal{UMF}(S)$ be the space obtained as the quotient of $\mathcal{MF}(S)$ obtained by identifying two elements of $\mathcal{MF}$ whenever these elements can be represented by topologically equivalent foliations, that is, forgetting the transverse measure.

We note that Masur and Minsky showed that the complex $C(S)$, equipped with its natural simplicial metric, is Gromov hyperbolic (cf. [46]), and that Klarreich identified the Gromov boundary of $C(S)$ as the subspace of $\mathcal{UMF}$ consisting of equivalence classes of minimal foliations (that is, foliations in which every leaf, including the singular ones, is dense) (cf. [35]).

We call $\mathcal{UMF}$ the space of unmeasured foliations on S.

The extended mapping class group $\Gamma^*(S)$ acts naturally by homeomorphisms on $\mathcal{UMF}$.

The space $\mathcal{UMF}$, equipped with the quotient of the topology of $\mathcal{MF}$, is non-Hausdorff. We shall analyze precisely this non-Hausdorffness, and we shall exploit it in the proof of the Theorem 19.1 on the rigidity of the action of Γ^* on $\mathcal{UMF}$.

We denote by $\mathrm{Homeo}(\mathcal{UMF})$ the group of homeomorphisms of $\mathcal{UMF}$.

The following result says that in some sense the action on $\mathcal{UMF}$ of the group $\mathrm{Homeo}(\mathcal{UMF})$ coincides on a dense subset of $\mathcal{UMF}$ with the action of the extended mapping class group $\Gamma^*(S)$ on that space.

Theorem 19.1 (cf. [59])**.** *Let S be a closed surface of genus ≥ 2. Then, there exists a dense subset $\mathcal{D}$ of $\mathcal{UMF}$ which is invariant by the group* $\mathrm{Homeo}(\mathcal{UMF})$ *and such that if h is any homeomorphism of $\mathcal{UMF}$, then there exists an element h^* of $\Gamma^*(S)$ such that the restriction on $\mathcal{D}$ of the actions of h and h^* coincide.*

Suppose furthermore that S is not the closed surface of genus 2. Then, if h_1 and h_2 are distinct elements of $\Gamma^*(S)$, their induced actions on $\mathcal{D}$ are different. In particular, the natural homomorphism from $\Gamma^*(S)$ to $\mathrm{Homeo}(\mathcal{UMF})$ *is injective. In the case of genus two, the kerbel of the homomorphism from Γ^* to the homeomorphism group of $\mathcal{D}$ is $\mathbb{Z}_2$.*

The set $\mathcal{D}$ in the statement consists of the natural image in $\mathcal{UMF}$ of the set $\mathcal{S}'$ of systems of curves on S.

We shall give all the ingredients of the proof of Theorem 19.1.

The proof involves the notion of adherence, and we start by defining this notion.

Let X be a topological space.

Definition 19.2 (Adherence)**.** Let x and y be two points in X. We say that x is *adherent to* y in X if every neighborhood of x intersects every neighborhood of y.

Definition 19.3 (Adherence set)**.** Let x be a point in X. The *adherence set* of x is the set of elements in X which are adherent to x.

Definition 19.4 (Complete adherence set)**.** A subset Y of X is a *complete adherence set* in X if for any two elements x and y of Y, x is adherent to y in X.

Definition 19.5. Let x be a point in X. The *adherence number* $\mathcal{N}(x)$ of x is the element of $\mathbb{N} \cup \{\infty\}$ defined as

$$\mathcal{N}(x) = \sup\{\mathrm{Card}(A) \mid x \in A \text{ and } A \text{ is a complete adherence set in } X\}.$$

Given two partial measured foliations F and G on S with disjoint supports, their union can be naturally considered as a (partial) measured foliation on S, which we shall denote by $F + G$. We shall say that F is a *subfoliation* of $F + G$ and that the foliation $F + G$ (or any foliation equivalent to $F + G$) *contains* the foliation F (or any foliation equivalent to F).

Lemma 19.6. *Let F and G be two measured foliations on S. Then, the following are equivalent:*

(1) $i(F, G) = 0$.
(2) $F \sim F'$ and $G \sim G'$, where F' and G' are partial measured foliations on S such that $F' = F_1 + F_2$ and $G' = G_1 + G_2$ where F_1 and G_1 are equal as topological foliations, and where F_2 and G_2 have disjoint supports. (Some of the partial foliations F_1, F_2, G_1 and G_2 may be empty.)
(3) $[F]$ is in the adherence set of $[G]$ in $\mathcal{UMF}$.

Proof. The equivalence between (1) and (2) is well-known. Let us prove that (2) implies (3). Let F' and G' be partial measured foliations as in (2). For the proof, we can assume that all four measured foliations, F_1, F_2, G_1 and G_2, are not empty. (In the contrary case, the proof is even simpler.) We then consider the measured foliation $F' + G_2$, and we let $[F' + G_2]$ be its equivalence class in $\mathcal{MF}$. For any sequence t_n of positive numbers convering to 0, the sequences $[F' + t_n G_2]$ in $\mathcal{MF}$ converges to $[F']$. For the corresponding elements in $\mathcal{UMF}$, we have $[F' + t_n G_2] = [F' + G_2]$ for all n. This shows that in $\mathcal{UMF}$, $[F' + G_2]$ is in every neighborhood of $[F']$ (recall that a set in $\mathcal{UMF}$ is open if and only if its inverse image in $\mathcal{MF}$ is open). By the same argument, $[G' + F_2]$ is in every neighborhood of $[G']$ in $\mathcal{UMF}$. Since $[G' + F2] = [F' + G_2]$ in $\mathcal{UMF}$, $[F']$ is adherent to $[G']$. We now prove that (3) implies (1). The proof is by contradiction. Suppose that $i(F, G) \neq 0$. Then, by the continuity of the intersection function, we can find neighborhoods $N([F])$ of $[F]$ in $\mathcal{MF}$ and $N([G])$ of $[G]$ in $\mathcal{MF}$ such that $i(x, y) \neq 0$ for all x in $N([F])$ and for all y in $N([G])$. We can furthermore suppose that $N([F])$ and $N([G])$ are saturated sts with respect to the equivalence relation on $\mathcal{MF}$

which identifies two equivalence classes of measured foliations whenever they can be represented by the same topological foliation. The images of $N([F])$ and of $N([G])$ in $\mathcal{UMF}$ are disjoint neighborhoods of the images of F and of G in that space. This shows that $[F]$ is not in the adherence set of $[G]$ in $\mathcal{UMF}$. $\square$

We shall use the following proposition, which is a direct consequence of the equivalence (1)⇔(2) of Lemma 19.6.

Proposition 19.7. *Let F be a measured foliation and let $[F]$ be its image in $\mathcal{UMF}$. Then, the adherence set of F is the set of equivalence classes in $\mathcal{UMF}$ of foliations G which are of the form $G_1 + G_2$ where G_1 is a sum of components of F and where G_2 is a partial measured foliation whose support is disjoint from the support of a representative of F by a partial measured foliation.*

We denote by $j : \mathcal{S}' \hookrightarrow \mathcal{UMF}$ the natural inclusion induced from the map that we considered in Section 17 above that associates to each weighted system of curves the corresponding annular measured foliation.
We shall also call a foliation on S representing an element of $\mathcal{UMF}$ which is the image of some element of $\mathcal{S}'$ by the map j an *annular foliation*.

Proposition 19.8. *Let F be a measured foliation on S and let $[F]$ denote the corresponding element of $\mathcal{UMF}$. Then, $\mathcal{N}([F]) = 2^q - 1$, where q is the maximum, over all measured foliations G containing F, of the number of components of G.*

Proposition 19.9. *If $x \in \mathcal{UMF}$ is the class of an annular foliation, then $\mathcal{N}(x) = 2^q - 1$, with $q = 3g - 3$. Furthermore, if $y \in \mathcal{UMF}$ is not the class of an annular foliation, then $\mathcal{N}(y) < \mathcal{N}(x)$.*

Proof. This follows from Proposition 19.8, and from the fact that if F is annular, then the maximal number of components of a measured foliation containing it is $3g - 3$, and if F is not annular, the maximal number of components of a measured foliation G containing it is $< 3g - 3$. $\square$

The following is a consequence of Proposition 19.9 and of the fact that any homeomorphism of $\mathcal{UMF}$ preserves adherence numbers of points.

Corollary 19.10. *Any homeomorphism of $\mathcal{UMF}$ preserves the image of $\mathcal{S}'$ in $\mathcal{UMF}$ by the map j.*

Let us now draw a another consequence of Proposition 19.9 which will be useful in the proof of Proposition 19.12 below.

Corollary 19.11. *If $g \neq g'$, then $\mathcal{UMF}(S_g)$ is not homeomorphic to $\mathcal{UMF}(S_{g'})$.*

As in the preceding section, for every $k \geq 1$, we denote by of $\mathcal{S}_k$ the subset of $\mathcal{S}'$ consisting of homotopy classes that are representable by pairwise disjoint and non-homotopic k simple closed curves.

Proposition 19.12. *For any $k \geq 1$, any homeomorphism of $\mathcal{UMF}$ preserves the image of $\mathcal{S}_k$ by the map $j : \mathcal{S}' \to \mathcal{UMF}$.*

Proof. Let f be a homeomorphism of $\mathcal{UMF}$. By Corollary 19.10, f preserves the subset $j(\mathcal{S}')$ of $\mathcal{UMF}$. Now $j(\mathcal{S}')$ is the disjoint union of the spaces $j(\mathcal{S}_k)$ with $k = 1, \ldots 3g-3$. It suffices to prove that if $m \neq n$ and for any $[F] \in j(\mathcal{S}_m)$ and $[G] \in j(\mathcal{S}_n)$, we have $f([F]) \neq [G]$. By Proposition 19.7, the adherence set $\mathcal{A}([F])$ of $[F]$ (respectively $\mathcal{A}([G])$ of $[G]$) in $\mathcal{UMF}$ is homeomorphic to a finite union of spaces which are all homeomorphic to a space $\mathcal{UMF}(S')$ (respectively $\mathcal{UMF}(S'')$) where S' and S'' are (not necessarily connected) subsurfaces of S which are the complement of the support of partial foliations F and G respectively, representing $[F]$ and $[G]$ respectively. Since the number of components of $[F]$ and $[G]$ are distinct, then if g' and g'' are respectively the genera of S' and S'', we have $g' \neq g''$. By Corollary 19.11, $\mathcal{A}([F])$ is not homeomorphic to $([G])$. Thus, f cannot send $[F]$ to $[G]$, which completes the proof of the proposition. $\square$

Now we can prove Theorem 19.1.
Let $\mathcal{D} = j(\mathcal{S}') \subset \mathcal{UMF}$ and let h be a homeomorphism of $\mathcal{UMF}$. Since h preserves the set $j(\mathcal{S}_1) = j(\mathcal{S})$, h induces a map from the vertex set $j(\mathcal{S})$ of $C(S)$ to itself. Since h is a homeomorphism, the map on $C(S)$ is bijective. Furthermore, since, for each $k \geq 2$, h preserves the set $j(\mathcal{S}_k)$ in $\mathcal{UMF}$, this action on the vertex set of $C(S)$ can be naturally extended to a simplicial automorphism h' of $C(S)$. If S is not the closed surface of genus two, it follows from the theorem of Ivanov (Theorem 6.2) that the automorphism h' is induced by an element h'' of the extended mapping class group $\Gamma^*(S)$. The element h'' acts on $\mathcal{UMF}$, and, by construction, the action induced on $\mathcal{D}$ by this map is the same as that of h. This completes the proof of the existence part of Theorem 19.1. The result about the kernel is a consequence of the statement in Ivanov's theorem about the kernel of the homomorphism from the extended mapping class group to the automorphism group of the curve complex.

Part 4. AUTOMORPHISMS OF TEICHMÜLLER SPACES

The rigidity theorems that we present in the rest of these notes concern Teichmüller space equipped with its complex-analytic structure, with its Teichmüller metric and with its Weil-Petersson metric. They say that (except in the case of a small number of special surfaces) the automorphism group of Teichmüller space coincides with the extended mapping class group. This shows in particular that Teichmüller space is highly inhomogeneous, which is one of the features that make this space so interesting. For instance, the rigidity results imply that a Fenchel-Nielsen deformation, with a fixed twist angle, does not induce a holomorphic automorphism, neither an isometry for the Teichmüller metric or of the Weil-Petersson metric, except in the case of a full twist (where the Fenchel-Nielsen deformation is induced from the action of mapping class, viz. a Dehn twist).[5]

Let us note that various generalizations of the rigidity results that we present here, which apply to infinite-dimensional Teichmüller spaces, have been obtained by C. Earle and F. Gardiner (see [12]), by V. Markovic (see [42]) and by N. Lakic (see [37]); see also the expositions in [17] and [18].

Before stating the results, I will give a short exposition of classical Teichmüller theory.

20. Teichmüller space

We shall deal, as it is usually done in this setting, with punctured surfaces rather than with surfaces with boundary. Thus, $S = S_{g,n}$ denotes here an oriented connected surface of finite type, of genus $g \geq 0$ with $n \geq 0$ punctures. In particular, this will spare us the technicalities of conformal structures with boundary.

To avoid talking about very special cases, we shall assume that the surface S is not a sphere with 0, 1 or 2 punctures.

The Teichmüller space of S is a space of equivalence classes of conformal structures on S.

We recall that a *conformal structure* on S is a maximal atlas $\{(U_i, z_i)\}_{i\in\mathcal{I}}$ where, for each $i \in \mathcal{I}$, U_i is an open subset of S and z_i a homeomorphism from U_i to an open subset of the complex plane $\mathbb{C}$, satisfying $\bigcup_{i\in\mathcal{I}} U_i = S$, and such that any map of the form $z_i \circ z_j^{-1}$, defined on each conected component of $z_j(U_i \cap U_j)$, is the restriction of a holomorphic map of $\mathbb{C}$. We shall consider only conformal structures on S such that each puncture has a neighborhood which is biholomorphically equivalent to a punctured disk in $\mathbb{C}$.[6] By the classical "Removable Singularity Theorem", the last condition means that the conformal structure on S is obtained from a conformal structure on a closed surface of genus g by removing n points. Thus, a conformal structure in our sense can also be considered as a conformal structure on a closed surface, with a certain number of distinguished points.

We note that since any conformal map from $\mathbb{C}$ to itself is orientation-preserving, a conformal structure on S induces an orientation on that surface. We shall only consider conformal structures that induce on S the orientation that we started with.

A surface equipped with a conformal structure is called a *Riemann surface.*

We define a distance between conformal structures, and for that we use the notion of a quasiconformal homeomorphism between Riemann surfaces. Before talking about quasiconformal homeomorphisms, we recall that a *quadrilateral* in a Riemann surface S is an embedded closed disk with two distinguished disjoint arcs in its boundary. We call the distinguished arcs the

[5] It is known however that the Fenchel-Nielsen flow is real-analytic.

[6] Remember that in the neighborhood of a puncture, the surface could be, from the complex-analytic point of view, either a disk or an annulus.

vertical sides of the quadrilateral. Any quadrilateral Q in S is equipped with a conformal structure with boundary induced from that of S. By the Riemann Mapping Theorem, there exists a unique positive real number $\mathrm{Mod}(Q)$ and a conformal homeomorphism ϕ from Q to the rectangle R in the Euclidean plane $\mathbb{R}^2$ with vertices at $(0,0)$, $(\mathrm{Mod}(Q),0)$, $(\mathrm{Mod}(Q),1)$ and $(0,1)$, such that ϕ sends the vertical sides of Q to the vertical sides of R (that is, the sides of length 1, see Figure 16). The value $\mathrm{Mod}(Q)$ is called the *modulus* of Q (and of R).

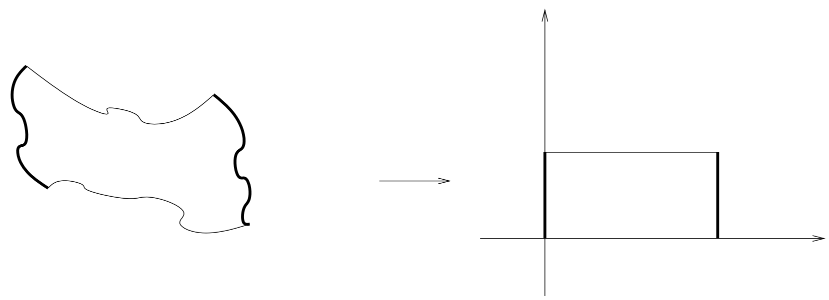

FIGURE 16. There is a conformal map from the quadrilateral Q on the left to the Euclidean rectangle on the right, sending the vertial sides of Q (which are drawn in bold lines) to the vertical sides of the Euclidean rectangle. The Euclidean rectangle is unique up to homothety.

Let G and H be Riemann surfaces and let $f : G \to H$ be a homeomorphism. The homeomorphism f transforms any quadrilateral in G into a quadrilateral in H, and it is said to be *quasiconformal* if we have

$$K(f) = \sup_Q \frac{\mathrm{Mod}(f(Q))}{\mathrm{Mod}(Q)} < \infty,$$

where the supremum is taken over all quadrilaterals Q in G. The value $K(f)$ is called the *quasiconformal dilatation* of f. For every $K \geq K(f)$, f is said to be K-quasiconformal homeomorphism. From the definition, it follows that the inverse of a K-quasiconformal homeomorphism is also a K-quasiconformal homeomorphism.

If f is conformal, it preserves the moduli of quadrilaterals, and its quasiconformal dilatation is equal to one. Conversely, a 1-quasi-conformal homeomorphism is a conformal homeomorphism. Thus, the quasiconformal dilatation of a map is a measure of the defect in conformality of that map.

We note that there are several other ways of defining quasiconformality. We refer to [3] and to [17] for the various definitions and for their equivalence. In particular, if the homeomorphism f is of class C^1, then, at any point of the surface, f transforms an infinitesimal circle into an infinitesimal ellipse, and a measure of the defect in conformality of f at this point is the ratio of the big axis to the small axis of the image infinitesimal ellipse. The quasiconformal dilatation of f is then equal to the supremum over the surface S of these ratios at every point.

Now we define the Teichmüller space of S. This definition involves the choice of a base Riemann surface, and Teichmüller space will be seen as a space of equivalence classes of quasiconformal homeomorphisms from this base Riemann surface to other Riemann surfaces. Although the fact that we have to make a choice of a base surface may seem unnatural, the complex-theoretic point of view of Teichmüller theory often involves such a choice. For instance, there is a very useful parametrization of Teichmüller space by the space of Beltrami differentials on the base Riemann surfaces. Likewise, the space of quadratic differentials on the base Riemann surface provides a useful ray structure on its Teichmüller space, and so on.

Thus, in this section, we consider that our surface S is a Riemann surface.

Definition 20.1 (Teichmüller space)**.** The *Teichmüller space* $\mathcal{T}(S)$ of S is the space of equivalence classes of pairs (S', f) where S' is a Riemann surface and $f : S \to S'$ is a quasiconformal homeomorphism and where two pairs (S_1, f_1) and (S_2, f_2) are considered to be equivalent if there exists a conformal homeomorphism $g : S_1 \to S_2$ such that the map $f_1 \circ f_2 : S \to S$ is homotopic to the identity.

A pair (S', f) is usually called a *marked Riemann surface*, and f (or its homotopy class) is called the *marking* of S.

We shall sometimes denote the equivalence class of (S', f) by $[S', f]$. The *basepoint* of Teichmüller space is the equivalence class of the pair (S, Id). The equivalence class of this basepoint will simply be denoted by $[S]$.

Of course, by pulling back the conformal structure on S' to a conformal structure on S, one can also see the elements of Teichmüller space as equivalence classes of conformal structures on the base surface S (and the marking is implicit).

There is a natural action of the mapping class group $\Gamma(S)$ on Teichmüller space. To see this action, one first needs to know that in any homotopy class of homeomorphisms of S, there is a quasiconformal homeomorphism. (Note that this uses the fact that the punctures of S are also punctures in the conformal sense, which was part of our definition of a conformal structure on S). Then, given an element γ in $\Gamma(S)$, it transforms the equivalence class of a pair (S', f) into the equivalence class of the pair $(S', f \circ g^{-1})$ where g is any quasiconformal homeomorphism of S in the class γ.

The quotient of the action of $\Gamma(S)$ on Teichmüller space is Riemann's moduli space.

In fact, there is an action of the extended mapping class group $\Gamma^*(S)$ on Teichmüller space, which is defined as follows. If γ is an element of $\Gamma^*(S)$ which is not in $\Gamma(S)$, we take an orientation-reversing quasiconformal homeomorphism g representing γ, and we define the image by γ of the equivalence class of a pair (S', f) to be the equivalence class of the pair $(\overline{S'}, j_S \circ f \circ g^{-1})$ where $\overline{S'}$ is the mirror-image of the Riemann surface S', that is, the Riemann

surface obtained by composing with complex conjugation each coordinate chart of the atlas defining the Riemann surface S', and where $j_S : S' \to \overline{S'}$ is the canonical map between a surface and its mirror image (see [1] p. 93). In loose terms, in the context of Teichmüller spaces, we can think of the extended mapping class group as the mapping class group extended by the involution which sends each Riemann surface to its mirror image. (To be more precise one should be careful about markings and orientations.)
By a result attributed to Fricke, the extended mapping class group acts properly discontinuously on Teichmüller space, and this action is faithful except in the cases $(g,n) = (0,3), (0,4), (1,1), (1,2)$ or $(2,0)$. The reason why this result fails for $(g,n) = (0,3)$ is that the Teichmüller space of $S_{0,3}$ is reduced to a point, whereas the mapping class group of that surface is not the trivial group. For the other exceptional surfaces, the reason is the existence of the hyperelliptic involutions, which act trivially on Teichmüller space.
We shall deal in the next sections with several structures of Teichmüller space, and we note right away that there is an instance where Teichmüller space can be easily described (without being reduced to a point), namely, the case where the surface S is the torus S_1. Indeed, a torus, equipped with a complex structures, can always be represented as the quotient of the complex plane $\mathbb{C}$ by a lattice generated by two translations $z \mapsto z+w_1$ and $z \mapsto z+w_1$, where w_1 and w_2 are two complex numbers that are independent over $\mathbb{R}$. By conjugating the lattice by multiplication by a complex number, we end up with representing in a unique manner any equivalence class of complex structures on the torus by a lattice generated by the two translations $z \mapsto z+1$ and $z \mapsto z+w$, where w is a complex number whose imaginary value is > 0. This gives a one-to-one correspondence between the Teichmüller space of the torus and the upper half-plane $\mathbb{H}$ in $\mathbb{C}$. It turns out that this parametrization of $\mathcal{T}(S_1)$ by the upper half-plane is exact in several respects. For instance, the complex analytic structure of Teichmüller space (which we shall discuss in the next section) coincides with the complex analytic structure of $\mathbb{H}$ induced from its inclusion in the complex plane, and the Teichmüller metric of the Teichmüller space (which we shall discuss in Section 22 below) coincides with the Poincaré hyperbolic metric on $\mathbb{H}$.
For surfaces of higher genus, there are no such simple descriptions of the complex analytic and the metric structures of their Teichmüller spaces.

21. Automorphisms of the complex structure of Teichmüller space

This section deals with the complex analytic structure of Teichmüller space. The bases of this theory are highly non-trivial, partly because they involve some deep and technical results in analysis. Therefore, this attempt to present this theory in a few lines in necessarily superficial, but we need to do it so that the theorems below are intelligible, and we hope that this presentation will incite the interested nonspecialist reader to go through the original papers.

Ahlfors and (independently) Rauch defined a natural complex analytic structure on Teichmüller space $\mathcal{T}(S_{g,n})$ which make this space a complex analytic manifold of complex dimension $3g-3+n$ and, more precisely, holomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+n}$.

There is a way of presenting the complex structure of Teichmüller space which is quite natural to state if one is satisfied with loose terms, and this is the following. It is well-known that each Riemann surface can be represented by a complex algebraic curve, more precisely, the surface can be described as the zero set of a polynomial in two variables with complex coefficients. By varying the coefficients, one can deform the Riemann surface, and this gives the idea of the existence of a complex structure on the space of Riemann surfaces. In fact, there is no marking involved in this point of view, and one describes in this way a complex structure on Riemann's moduli space rather than on Teichmüller space, which is fine since Teichmüller space is a branched covering of Riemann's moduli space, and therefore the complex structure on Riemann's moduli space lifts to a complex structure on Teichmüller space.

The complex structure that we describe next coincides with the one we just mentioned, although the proof of this fact is not an easy matter. Let us also note that the complex structure on Teichmüller space that we shall describe is natural in the sense that it makes the period functions of the abelian differentials on Riemann surfaces to be holomorphic functions on Teichmüller space.

The complex structure of Teichmüller space is usually described via a representation of Riemann surfaces by Beltrami differentials on the base Riemann surface. The complex structure on $\mathcal{T}(S_{g,n})$ is then induced from a natural complex structure of the space of Beltrami differentials on S, see e.g. [1].

We recall that a Beltrami differential μ on S is a tensor of type $(-1,1)$, that is, an invariant object which in the local holomorphic coordinates of S is of the form $\mu(z)d\overline{z}/dz$, where μ is an essentially bounded measurable function. The invariance means that if w is another local holomorphic coordinate and if in that coordinate the Beltrami differential is written as $\nu(w)d\overline{w}/dw$, then, at the overlap between the two coordinate charts, we have $\mu(z)d\overline{z}/dz = \nu(w)d\overline{w}/dw$.

We note that given such a Beltrami differential, the real-valued function $|\mu|$ which a priori is defined only in local coordiantes, is well defined on the surface S, and the essential boundedness of μ simply says that the function $|\mu|$ on S is essentially bounded.

It turns out that Teichmüller space can be considered as a space of Beltrami differentials on S up to a certain equivalence relation. The idea of representing conformal structures by Beltrami differentials is the following. Given a Beltrami differential μ on S, one obtains a new conformal structure by means of a quasiconformal mapping f which is a solution of the following differential equation, called the Beltrami equation with coefficient μ,

$$f_{\overline{z}} = \mu f_z.$$

In other words, the new conformal structure is obtained by composing the coordinate charts of the original structure with maps induced by this quasiconformal homeomorphism f. The Beltrami differential μ is called the *complex dilatation* of f. The invariance property of μ ensures that the equation $f_{\overline{z}} = \mu f_z$, which is a priori defined in coordinate charts, is well defined on the surface S.

There is a natural notion of equivalence between Beltrami differentials, which is defined in such a way that equivalent conformal structures on S are represented by equivalent Beltrami differentials.

The basepoint S is obtained by taking $\mu = 0$, since the equation $f_{\overline{z}} = 0$ implies that f is conformal.

The space of Beltrami differentials on S has a natural structure of a Banach complex space, and this structures induces the complex analytic structure on $\mathcal{T}$.

Now we must say a few words about quadratic differentials.

We recall that a holomorphic quadratic differential q on the base Riemann surface S is a tensor of type $(2,0)$, that is, an invariant object that has an expression $q(z)dz^2$ in each holomorphic local chart, where $q(z)$ is a holomorphic function of z, the holomorphic local coordinate in that chart. Invariance of q means that if w is the local coordinate in another chart and if a local expression of q in that chart is $p(w)dw^2$, then at the overlap between the two charts we have $q(z)dz^2 = p(w)dw^2$ or, equivalently, $q(z)(dz^2/dw^2) = p(w)$.

The norm of a quadratic differential q on S is defined as the surface integral

$$\|q\| = \int_S |q(z)||dzd\overline{z}|.$$

(It is a consequence of the definition of a quadratic differential that the expression under the integral sign makes sense in local coordinates).

A quadratic differential is said to be in L^1 if its norm is finite. A holomorphic quadratic differential is in L^1 if and only if it is meromorphic at the punctures (or, rather, on the Riemann surface obtained by filling in the punctures) and has at worst simple poles there.

The vector space $Q(S)$ of L^1 holomorphic quadratic differentials on S is a complex Banach space, and by the Riemann-Roch therem, its dimension is $3g-3+n$. (This uses the fact that the quadratic differential is meromorphic with at worst simple poles at the punctures). There is an important holomorphic embedding of Teichmüller space into $Q(S)$, which was discovered by Bers, and whose definition uses the notion of Schwarzian derivative. We recall that given a locally injective holomorphic map $f : \Omega \to \mathbb{CP}^1$, where Ω a domain contained in the Riemann sphere $S^2 \simeq \mathbb{CP}^1$, its Schwarzian derivative Sf is given by the formula

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f'}\Big)^2.$$

There are several ways in which the notion of Schwarzian derivative is well-behaved under Möbius transformation. First of all, in some precise sense, Sf is a measure of how far f is from being the restriction of a Möbius transformation. Furthermore, given a subgroup G of $\mathrm{PSL}(2,\mathbb{C})$ acting properly discontinuously on Ω, the Schwarzian derivative Sf gives a well-defined quadratic differential on the quotient Riemann surface Ω/G. We refer the reader to the book by Lehto [38] and to the paper by Thurston [68] for information on this subject. The image of $\mathcal{T}(S_{g,n})$ by its embedding into $Q(S)$ is a bounded open subset of $Q(S)$, see e.g. the books by Ahlfors [1] and by Fletcher and Markovic [17].
There is an identification between the holomorphic tangent space to Teichmüller space at any point x in that space and a vector space of Beltrami differentials on a surface X representing x, modulo the Beltrami differentials that are "infinitesimally trivial", that is, Beltrami differentials that are tangent to trivial infinitesimal deformations. It turns out that a Beltrami differential μ is trivial if and only if it satisfies $\int_X \mu q = 0$ for all holomorphic quadratic differentials q on the X.
The cotangent space to $\mathcal{T}(S_{g,n})$ at x is identified with the space of L^1 holomorphic quadratic differentials, with a natural pairing between tangent and cotangent spaces given by

$$(\mu, q) \mapsto \int_X \mu q.$$

We note that since in local coordinates, μ has the expression $\mu(z)d\overline{z}/dz$ and since q has an expression $q(z)dz^2$, than μq has the expression $\mu(z)q(z)dzd\overline{z}$, that is, $\mu(z)q(z)|dz|^2$, which is an object we can integrate on the surface X. This identification of the cotangent bundle of Teichmüller space is important because it turns out that the Teichmüller and the Weil-Petersson metrics are best decribed by co-norms on the cotangent spaces, rather than by norms on tangent spaces. We shall see this fact more precisely in the next two sections.
The extended mapping class group $\Gamma^*(S)$ acts on Teichmüller space equipped with its complex structure as a group of biholomorphic or anti-biholomorphic homeomorphisms. More precisely, an element of the mapping class group acts as a biholomorphic homeomorphism of $\mathcal{T} = \mathcal{T}(S_{g,n})$ (note that a bijective holomorphic map is necessarily biholomorphic), and an element of the extended mapping class group which is not in the mapping clas group acts as an anti-holomorphic homeomorphism.
Let $\mathrm{Holo}(\mathcal{T})$ be the group of holomorphic automorphisms of $\mathcal{T}$ and let $\mathrm{Holo}^*(\mathcal{T})$ be the group of holomorphic or anti-holomorphic automorphisms of $\mathcal{T}$.

In 1971, H. L. Royden proved that for closed surfaces of genus ≥ 2, the natural map $\Gamma(S) \to \mathrm{Holo}(\mathcal{T})$ is onto (cf. Royden [62]). Earle and Kra [14] extended Royden's result and they studied all surfaces with punctures. We summarize these rigidity result in the following:

Theorem 21.1 (cf. Royden [62] for the case of closed surfaces, and Earle and Kra [13] for the remaining cases)**.** *Let S be a surface which is not a sphere with at most four punctures, a torus with at most two punctures or a closed surface of genus 2. Then, the natural homomorphisms*

$$\Gamma^*(S) \to \mathrm{Holo}^*(\mathcal{T})$$

and

$$\Gamma(S) \to \mathrm{Holo}(\mathcal{T})$$

are isomorphisms.

The proofs of these results that were given by Royden and by Earle and Kra are based on the fact that the holomorphic and anti-holomorphic automorphisms of $\mathcal{T}$ are the isometries for the Teichmüller metric on that space, and therefore, the rigidity results follow from the corresponding rigidity results for the Teichmüller metric. The Teichmüller metric is a Finsler metric and the proof of the rigidity result for this metric is based on an analysis of the smoothness of the unit sphere of that metric at each tangent space (see Section 22 below).

In particular, for the cases that are excluded in the statement of Theorem 21.1, the situation is the same as the one of the isometry group of Teichmüller metric, which we summarize in Theorem 22.2 below.

As these remarks suggest, the results follow from local results. In fact, Earle and Kra proved the following stronger rigidity result that concerns local holomorphic maps between Teichmüller spaces. Let $S_{g,n}$ and $S_{g',n'}$ be two surfaces neither of which is a sphere with at most four punctures, a torus with at most two punctures or a closed surface of genus 2 and suppose that there exists a holomorphic map f from an open set U of $\mathcal{T}(S_{g,n})$ onto an open set of $\mathcal{T}(S_{g,n})$. Then, $(g,n) = (g',n')$ and the map f is the restriction to U of the image of an element of the extended mapping class group of $S_{g,n}$ in the corresponding group of biholomorphic maps of Teichmüller space.

This result generalizes a theorem by D. B. Patterson [60] which says that if $(g,n) \neq (g',n')$ there cannot exist any biholomorphic homeomorphism between the Teichmüller spaces $\mathcal{T}(S_{g,n})$ and $\mathcal{T}(S_{g',n'})$, except for the following biholomorphic homeomorphisms:

$$\mathcal{T}(S_{2,0}) \simeq \mathcal{T}(S_{0,6}),$$

$$\mathcal{T}(S_{1,2}) \simeq \mathcal{T}(S_{0,5}),$$

$$\mathcal{T}(S_{1,0}) \simeq \mathcal{T}(S_{1,1}) \simeq \mathcal{T}(S_{0,4}).$$

In particular, there are Teichmüller spaces which have the same dimension that are not biholomorphically equivalent.

A basic ingredient in the approaches of Royden and of Earle-Kra is the fact that the Teichmüller metric of Teichmüller space coincides with the Kobayashi metric of that space. Kobayashi metrics have the property that the biholomorphic and anti-biholomorphic homeomorphisms of the spaces on which they are defined are isometries of these metrics. In particular, the biholomorphic and anti-biholomorphic homeomorphisms of $\mathcal{T}(S_{g,n})$ are isometries of the Kobayashi metric. Thus, to study the group of biholomorphic and anti-biholomorphic homeomorphisms of Teichmüller space, one is reduced to study isometries of its Kobayashi metric. Royden and Earle-Kra proved that the natural homomorphism from the extended mapping class group in the group of isometries of the Teichmüller metric is an isomorphism (except in the excluded cases mentioned above). We shall discuss this in the next section.

22. Isometries of the Teichmüller metric

We first recall the definition of the Teichmüller metric.

Definition 22.1 (The Teichmüller metric)**.** The *Teichmüller metric* on $\mathcal{T}_{g,n}$ is the function $d_T : \mathcal{T}_{g,n} \times \mathcal{T}_{g,n} \to \mathbb{R}$ which associates to each pair of equivalence classes of marked Riemann surfaces $G = [S_1, f_1]$ and $H = [S_2, f_2]$ the quantity

$$d_T(G, H) = \frac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal homeomorphisms $f : S_1 \to S_2$ such that f_2 is homotopic to $f \circ f_1$.

We now recall the definition of the Kobayashi pseudo-metric[7] d_K associated to a complex manifold X.
Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ equipped with its Poincaré metric.
One first defines a function $d'_K : X \times X \to \mathbb{R}$ by the formula

$$d'_K(x, y) = \inf_{f,a,b} d_{\mathbb{D}}(a, b)$$

for x and y in X, the infimum being taken over all holomorphic maps $f : \mathbb{D} \to X$ and over all points a and b in $\mathbb{D}$ satisfying $f(a) = x$ and $f(b) = y$. The map d'_K does not necessarily satisfy the triangle inequality, and the Kobayashi semi-metric d_K is defined as the largest semi-metric on X satisfying $d_K \leq d'_K$.
An important breakthrough in Teichmüller theory was the result obained by Royden in his paper [62], saying that the Teichmüller metric on $\mathcal{T}$ coincides with the Kobayashi metric of that space.
We already noted that in the case where the surface S is a torus, its Teichmüller space, equipped with its complex analytic structure and with its

[7]A pseudo-metric satisfies all the axioms of a metric except the axiom $d(x, y) = 0 \Rightarrow x = y$.

Teichmüller metric, is isomorphic to the unit disk $\mathbb{D}$ equipped with its standard complex structure and with its Poincaré metric respectively. Thus, using Royden's result, one can imagine the Teichmüller space of an arbitrary surface as a space which in some sense is filled with Teichmüller spaces of tori. This remark is at the basis of the theory of Teichmüller disks in Teichmüller spaces, that is, the theory of holomorphic isomorphic embeddings of the Poincaré disk in these spaces; see for instance [44] and [25].
It follows easily from the definition that a holomorphic (or anti-holomorphic) homeomorphism of a complex manifold X acts by isometries with respect to the Kobayashi metric. In particular, the extended mapping class group acts as an isometry group of the Kobayashi metric of Teichmüller space.
In the paper [62], Royden proved that for any closed surface S of genus ≥ 3, the natural map from the extended mapping class group $\Gamma^*(S)$ into the isometry group $\mathrm{Isom}(\mathcal{T}(S))$ of the Teichmüller space of S equipped with the Kobayashi metric is an isomorphism, and that in the case where S is the closed surface of genus two, the natural map $\Gamma^*(S) \to \mathrm{Isom}(\mathcal{T}(S))$ is surjective, with the kernel being of order two and generated by the hyperelliptic involution.
Since the Teichmüller metric coincides with the Kobayashi metric, the result holds with the Kobayashi metric replaced by the Teichmüller metric.
Earle and Kra gave a proof of the corresponding results for surfaces with punctures. The general result on the isometry groups of Teichmüller spaces equipped with the Teichmülller metric is summarized in the following theorem:

Theorem 22.2 (Royden [62] for the case of closed surfaces, and Earle and Kra [13] for the remaining cases)**.** *Let S be a surface which is not a sphere with at most four punctures, a torus with at most two punctures, or a closed surface of genus two. Then, the natural homomorphism*

$$\Gamma^*(S) \to \mathrm{Isom}(\mathcal{T}(S))$$

is an isomorphism.
In the excluded cases, the situation is as follows:

(1) *If $S = S_{0,3}$, then $\mathcal{T}(S))$ is reduced to a point,* $\mathrm{Isom}(\mathcal{T}(S_{0,3})) = \{\mathrm{Id}\}$, *whereas $\Gamma^*(S_{0,3})$ is nontrivial (it is an order-two extension of the permutation group on three elements).*
(2) *If $S = S_{1,1}$, $S_{1,0}$ or $S_{0,4}$, then $\mathcal{T}(S))$ is isometric to the hyperbolic plane, and* $\mathrm{Isom}(\mathcal{T}(S_{1,1})) = \mathrm{Isom}(\mathcal{T}(S_{0,4}) = \mathrm{Mob}^*_{\mathbb{R}}$, *which is a 3-dimensional Lie group isomorphic to the isometry group of the upper half-plane $\mathbb{H}^2$ equipped with its Poincaré metric. In other words,* $\mathrm{Mob}^*_{\mathbb{R}}$ *is the group of transformations of $\mathbb{H}^2$ that are either of the form $z \mapsto (az+b)/(cz+d)$ with $ad-bc=1$ or $z \mapsto (a\overline{z}+b)/(c\overline{z}+d)$ with $ad-bc=-1$, where a,b,c,d are real coefficients. On the other hand, the extended mapping class group in these cases is respectively the group* $\mathrm{SL}(2,\mathbb{Z})$ *and a finite extension of that group.*

(3) *If $S = S_{1,2}$, then $\mathcal{T}(S)$ is isometric to the Teichmüller space of the sphere with 5 punctures $S_{0,5}$. It is known that $\Gamma^*(S_{1,2})$ is not isomorphic to $\Gamma^*(S_{0,5})$. Thus, the homomorphism $\Gamma^*(S_{1,2}) \to \mathrm{Isom}(\mathcal{T}(S_{1,2}))$ is not an isomorphism.*

(4) *The natural map $\Gamma^*(S_{2,0}) \to \mathrm{Isom}(\mathcal{T}(S_{2,0}))$ is onto, and its kernel is the order-two subgroup of $\Gamma^*(S_{2,0})$ generated by the hyperelliptic involution. Thus, in this case $\mathrm{Isom}(\mathcal{T}(S_{2,0}))$ is isomorphic to $\Gamma^*(S_{2,0})/\mathbb{Z}_2$.*

As in the case of biholomorphic homeomorphisms that we mentioned in Section 21, Earle and Kra proved in [13] the following local rigidity result which is stronger than Theorem 22.2. Given two surfaces $S_{g,n}$ and $S_{g',n'}$ which both are not a sphere with at most four holes, a torus with at most three holes or a cosed surface of genus two, if we have an isometry between an open set of $\mathcal{T}(S_{g,n})$ and an open set of $\mathcal{T}(S_{g',n'})$, then $(g,n) = (g',n')$ and the isometry is the restriction of the action of an element of the extended mapping class group of $S_{g,n}$ on the corresponding Teichmüller space.

In particular, there exist Teichmüller spaces which have the same dimension that are not isometric.

The Teichmüller metric is a Finsler metric, whose co-norm in the cotangent space to Teichmüller space at each point is the L^1-norm on the vector space of quadratic differentials at that point (see [62]). As we already mentioned, the proof of Theorem 22.2 is a local one, and the key point in that proof is a careful analysis of the smoothness (or, rather, the lack of smoothness) of the unit sphere in the vector space of L^1-integrable holomorphic quadratic differentials on S.[8]

The fact that the Teichmüller metric is a Finsler metric was first proved by B. O'Byrne (see [54]) using earlier work of Earle and Eells ([11]).

Finally, we mention that Ivanov gave a new proof of Royden's theorem that is based on the rigidity of the action of the extended mapping class goup on the curve complex (see [31]). The outline of Ivanov's approach is the following.

Let $h : \mathcal{T} \to \mathcal{T}$ be an isometry of the Teichmüller metric, and let x be an arbitrary point in $\mathcal{T}$. Then, h induces a homeomrophism between the set of geodesic rays in $\mathcal{T}$ starting at a point and the set of geodesic rays starting at the image point. Now there is a natural homeomorphism between the set of Teichmüller geodesic rays starting at a point and the space of equivalence classes of measured foliations on a Riemann surface representing that point. Indeed, the set of geodesic rays starting at the equivalence class of a Riemann surface is naturally parametrized by the vector space of quadratic differentials on that surface, and by a theorem of Hubbard and Masur, each equivalence class F of measured foliations determines in a unique way a quadratic differential whose horizontal foliations is in the class

[8] Note that despite the fact that the unit balls of the Teichmüller Finsler metric are not smooth, the distance function that is associated to this metric is smooth, see [10].

F. In this way, the isometry : $\mathcal{T} \to \mathcal{T}$ defines a homeomorphism of the space $\mathcal{MF}$. By an analysis involving the codimension sets of geometric intersection functions of measured foliations, Ivanov proves that this homeomorphism preserves the natural image of $\mathbb{R}_+^* \times \mathcal{S}$ in $\mathcal{MF}$, and hence, it defines a homeomorphism of $\mathcal{PMF}$ which preserves the image of the set $\mathcal{S}$ in that space. Therefore, one obtains a self-map of the vertex set of the curve complex $C(S)$ of the surface S. Ivanov proves that this automorphism of $C(S)$ preserves the set of pairs of edges that are connected by an edge. Using the fact that $C(S)$ is a flag complex, this implies that the map on the vertex set induces a simplicial automorphism of $C(S)$. By the Thorem of Ivanov-Korkmaz and Luo (Theorem 6.3, this automorphism is induced by an extended mapping class, and Ivanov proves that this extendd mapping class induces the isometry h of Teichmüller space that we started with.

23. Isometries of the Weil-Petersson metric

We already noted that if S is a Riemann surface and if $x = [S]$ is the corresponding point in Teichmüller space $\mathcal{T}$, then there is an identification of the cotangent space $T_x^*\mathcal{T}$ with the space of integrable holomorphic quadratic differentials on S that have at most simple poles at the punctures.

The Weil-Petersson product is the L^2 Hermitian product on the space of quadratic differentials on a given Riemann surface, defined by

$$\langle \phi, \psi \rangle = \int_S \frac{\phi\overline{\psi}}{\rho} dz \wedge d\overline{z}$$

where ρ is the line element of the unique complete finite volume hyperbolic metric on S (that is, ρ is the function in the local conformal coordinates z characterized by the fact that the line element of the hyperbolic metric has the form $ds^2 = \rho(z)|dz|^2$).

This gives a product on the cotangent space to Teichmüller space, which defines the Hermitian *Weil-Petersson metric* of $\mathcal{T}(S)$, by transporting the L^2 inner product from the cotangent space $T_x^*\mathcal{T}$ to the tangent space $T_x\mathcal{T}$ via the pairing $\langle \mu, \phi \rangle = \int_S \mu\phi$ between the tangent and cotangent spaces that we recalled above.

The Weil-Petersson metric is Kähler, non complete, with negative sectional curvature, and it is invariant under the action of the extended mapping class group Γ^* on $\mathcal{T}$.

In the paper [47], Masur and Wolf proved the following result

Theorem 23.1. *Let $S_{g,n}$ be a surface which is not a sphere with at most four holes or a torus with at most two holes. Then, the natural map*

$$\eta : \Gamma^*(S_{g,n}) \to T(S_{g,n})$$

is surjective.

In other words, every isometry of the Weil-Petersson metric is induced by an element of the extended mapping class group.

The approach that Masur and Wolf used for the proof of this theorem is analogous to the one used by Ivanov for his proof of Royden's theorem on the isometries of the Teichmüller metric that we recalled in Section 22. They associate to an arbitrary isometry h of the Weil-Petersson metric an automorphism of the curve complex of S. Then, they use the theorem of Ivanov-Korkmaz-Luo (Theorem 6.3) saying that (except in some special cases) the natural homomorphism from the extended mapping class group into the automorphism group of the curve complex is an isomorphism. Finally, they show that the action of this extended mapping class group on Teichmüller space is the one induced by the isometry h of the Weil-Petersson metric.

Masur and Wolf define the automorphism of the curve complex as follows. By previous work of Masur [45], the Weil-Petersson completion of the Teichmüller space $\mathcal{T}(S)$ is obtained by adding a frontier which is the union of the various Teichmüller spaces of surfaces with nodes or punctures obtained from S by pinching systems of curves to points. Each of these "boundary Teichmüller spaces" is equipped with its own Weil-Petersson metric and, equipped with that metric, each boundary Teichmüller space is isometrically embedded in the boundary of the completion of $\mathcal{T}(S)$. Now Masur and Wolf show that each isometry h of Teichmüller space extends to the Weil-Petersson completion of that space, acting by isometries on the Weil-Petersson metric of the boundary spaces. Since the boundary Teichmüller spaces are parametrized by systems of curves on the surface, this gives the desired action on the curve complex.

To show that the action of the mapping class h' obtained through the action on the curve complex induces the isometry h of the Weil-Petersson metric, Masur and Wolf analyze the actions of h and h' on the set of Weil-Petersson geodesics of Teichmüller space. Their arguments involve the CAT(0) geometry of the Weil-Petersson metric and a result of Wolpert on the convexity of the geodesic length function along Weil-Petersson geodesics.

We also note that in [71], Wolpert gave a proof of Theorem 23.1 which is a simplification of the proof by Masur and Wolf, and which is based on the fact (due to Wolpert) that the Weil-Petersson completion $\overline{\mathcal{T}(S)}$ is the convex hull of its maximally degenerate surfaces, i.e. the surfaces with a maximal number of nodes. These are the surfaces obtained from S by pinching a collection of curves that decomposes this surface into generalized pairs of pants (that is, spheres with three holes where each hole can be either a puncture or an open disk removed).

Finally, we note that Brock and Margalit gave in [7] another proof of the result of Masur and Wolf, which is based on the action of the isometry group of the Weil-Petersson metric on the pants decomposition graph of S. This action is obtained by making the Weil-Petersson isometry group act on the maximally degenerate surfaces. These surfaces are isolated points in the stratification of the completion $\overline{\mathcal{T}(S)}$, since each such stratum parametrizes the Teichmüller space of a sphere with three punctures, and such a Teichmüller space is reduced to a point. Then, Brock and Margalit apply

Margalit's result ([43], cf. Theorem 7.3 above) on the automorphism group of the pants decomposition complex. In the same paper, Brock and Margalit completed the result of Masur and Wolf by treating the cases of special surfaces that are excluded in the theorem of Masur and Wolf. The general result on the isometries of the Weil-Petersson metric can be stated as follows:

Theorem 23.2 (Masur-Wolf and Brock-Margalit [47], [7])**.** *Let S be a surface which is not a sphere with at most four holes, a torus with at most two holes or a closed surface of genus 2. Then, the natural homomorphism $\eta : \Gamma^*(S) \to \mathrm{Isom}(T(S))$ is an isomorphis. Furthermore, the situation in the excluded case is as follows:*

(1) *if S is a closed surface of genus 2 or a torus with 1 or 2 punctures,* $\ker(\eta) = \mathbb{Z}_2$*;*
(2) *if S is a sphere with 4 punctures,* $\ker(\eta) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$*;*
(3) *if S is a sphere with 3 punctures,* $\ker(\eta) = \Gamma^*(S)$

24. Thurston's asymmetric metric

This section contains no result but an open question.

In this section, S is a surface of negative Euler characteristic which has $n \geq$ punctures.

We recall that the Teichmüller space $\mathcal{T} = \mathcal{T}(S)$ can also be defined as the space of isotopy classes of complete finite area hyperbolic metrics on S.

One can also make a definition by an atlas, as follows. A hyperbolic structure on S is a maximal atlas $\{(U_i, \phi_i)\}_{i\in\mathcal{I}}$ where for each $i \in \mathcal{I}$, U_i is an open subset of S and ϕ_i is a homemorphism from U_i onto an open subset of the hyperbolic plane $\mathbb{H}^2$, satisfying $\displaystyle\bigcup_{i\in\mathcal{I}} U_i = S$ and such that any map of the form $\phi_i \circ \phi_j^{-1}$ is, on each connected component of $\phi_j(U_i \cap U_j)$, the restriction of an orientation-preserving isometry of $\mathbb{H}^2$.

A surface equipped with a hyperbolic structure carries a metric obtained by taking on each chart domain U_i the pull-back by the map ϕ_i the metric on $\phi_i(U_i)$ induced from its inclusion in $\mathbb{H}^2$. The metrics obtained on the various U_i's give a consistent way of measuring lengths of paths in S, and the metric we consider on S is the associated length metric, that is, the distance between two points is defined as the infinmum of the lengths of continuous and piecewise C^1 paths joining these points. We only consider hyperbolic metrics that are complete and of finite area.

An asymmetric metric on a set X is a map $L : X \times X \to \mathbb{R}_+$ that satisfies all the axioms of a metric except the symmetry axiom, and such that the symmetry axiom is not satisfied, that is, there exist x and y in X such that $L(x, y) \neq L(y, x)$. Teichmüller space is equipped with an asymmetric metric that was defined by Thurston as follows.

Given be two hyperbolic structures g and h on S given a homeomorphism $\varphi : S \to S$ which is isotopic to the identity, the Lipschitz constant $\mathrm{Lip}(\varphi)$ of

φ is defined by the formula

$$\mathrm{Lip}(\varphi) = \sup_{x \neq y \in S} \frac{d_h\big(\varphi(x), \varphi(y)\big)}{d_g\big(x, y\big)}.$$

The infimum of this Lipschitz constant over all homeomorphisms φ in the isotopy class of the identity is denoted by

$$L(g, h) = \log \inf_{\varphi \sim \mathrm{Id}_S} \mathrm{Lip}(\varphi).$$

Making g and h vary in their respective homotopy classes does not change the quantity $L(g, h)$ and thus we obtain a function which is well defined on $\mathcal{T}(S) \times \mathcal{T}(S)$. This function satisfies the axioms of an asymmetric metric and we call it Thurston's asymmetric metric. We shall denote it by the same letter:

$$L : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}_+.$$

Thurston showed that the quantity $L(g, h)$ can also be computed by comparing lengths of closed geodesics in the same homotopy class, measured with the metrics g and h. More precisely, for any homotopy class α of essential simple closed curves on S, we consider the quantity

$$r_{g,h}(\alpha) = \frac{l_h(\alpha)}{l_g(\alpha)}$$

and we set

$$K(g, h) = \log \sup_{\alpha \in \mathcal{S}} r_{g,h}(\alpha).$$

It is easy to see that $K \leq L$. In his paper [66], Thurston proves that $K = L$. We ask the following question: Is the isometry group $\mathrm{Isom}(\mathcal{T}, K)$ of Thurston's asymmetric metric equal to the natural image of the extended mapping class group in $\mathrm{Isom}(\mathcal{T}, K)$?

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