Deforming commuting directions in infinite matrices

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Infinite Toda chain 1

- Particles on a straight line with nearest neighbour interaction:

- q_n is the displacement of the n-th particle, $n \in \mathbb{Z}$.
- Equations of motion in dimensionless form are described by

$$\frac{dq_n}{dt} = p_n \text{ and } \frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \quad n \in \mathbb{Z}.$$

- Put

$$a_n := \frac{1}{2} e^{-(q_n - q_{n-1})} \text{ and } b_n := \frac{1}{2} p_n.$$

Infinite Toda chain 2

- Introduce the $\mathbb{Z} \times \mathbb{Z}$-matrices L resp. P by

$$\begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{b_{n-1}} & a_n & 0 & \ddots \\ \ddots & a_n & \mathbf{b_n} & a_{n+1} & \ddots \\ & 0 & a_{n+1} & \mathbf{b_{n+1}} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{0} & -a_n & 0 & \ddots \\ \ddots & a_n & \mathbf{0} & -a_{n+1} & \ddots \\ & 0 & a_{n+1} & \mathbf{0} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

- Equations of motion equivalent to:

$$\frac{dL}{dt} = PL - LP = [P, L].$$

Outline of the talk

- Basics of $\mathbb{Z} \times \mathbb{Z}$-matrices
- Lower Triangular Hierarchies (LTH)
- A geometric construction of solutions of LTH
- Upper Triangular Hierarchies (UTH)
- A geometric construction of solutions of UTH
- Combining both type of hierarchies
- Solutions of the combined hierarchy

The geometric setting 1

- Let S^1 be the unit circle in the complex plane.
- Hilbert space $H = L^2(S^1, \mathbb{C}^k)$ with elements

$$h = \sum_{n \in \mathbb{Z}} a(n) z^n, \text{ where } a(n) \in \mathbb{C}^k \text{ for all } n \in \mathbb{Z}.$$

- $(\cdot \mid \cdot)$ standard inner product on $\mathbb{C}^k$. Inner product on H:

$$< \sum_{n \in \mathbb{Z}} a(n) z^n \mid \sum_{n \in \mathbb{Z}} b(n) z^n > := \sum_{n \in \mathbb{Z}} (a(n) \mid b(n)).$$

- $\{f_i \mid 0 \leq i \leq k-1\}$ standard basis of $\mathbb{C}^k$. Hilbert basis of H:

$$e_{s+kj} := f_s z^j.$$

- To $B \in B(H)$ associated $\mathbb{Z} \times \mathbb{Z}$-matrix w.r.t. this basis

$$[B] = ([B]_{(l,k)})$$

The geometric setting 2

- $H^{(i)}$ is the subspace of H spanned by the

$$\{f_s z^i \mid 0 \leq s \leq k-1\}.$$

- $p^{(i)}$ the projection $H \mapsto H^{(i)}$
- The space H decomposes as the direct sum

$$H = \oplus_{i \in \mathbb{Z}} H^{(i)}$$

- To $B \in B(H)$ is associated the block decomposition $B = (B_{ij})$, where $B_{ij} := p^{(i)} \circ B \mid H^{(j)}$.
- Corresponding matrix decomposition $[B] = ([B_{ij}])$ in $k \times k$-blocks.

The geometric setting 3

- The subspace H_j, $j \in \mathbb{Z}$ is defined by

$$H_j = \oplus_{i \leq j} H^{(i)}.$$

- $p_j := \oplus_{i \leq j} p^{(i)}$ is the orthogonal projection onto H_j.
- Decomposition of any element $b \in B(H)$ w.r.t. the splitting $H = H_j \oplus H_j^{\perp}$, namely

$$b = \begin{pmatrix} b_{++}(j)) & b_{+-}(j) \\ b_{-+}(j) & b_{--}(j) \end{pmatrix}.$$

Basic $\mathbb{Z} \times \mathbb{Z}$-matrices 1

- For $A \in \mathrm{gl}_k(\mathbb{C})$, multiplying from the left defines a bounded map $M_A : H \mapsto H$ with $\mathbb{Z} \times \mathbb{Z}$-matrix

$$[M_A] = i_k(A) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{A} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{A} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{A} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Commuting directions: $\{i_k(A) \mid A \in \mathfrak{h}\}$, where $\mathfrak{h}$ is the diagonal matrices with basis

$$E_\alpha = F_\alpha, \;\; (E_\alpha)_{\gamma\delta} = \begin{cases} 1 & \text{if } \gamma = \delta = \alpha \\ 0 & \text{in other cases} \end{cases}$$

Basic $\mathbb{Z} \times \mathbb{Z}$-matrices 2

- $M_z : H \mapsto H$ "multiplication with z" ,

$$M_z(\sum_{n\in\mathbb{Z}} a(n)z^n) = \sum_{n\in\mathbb{Z}} a(n)z^{n+1}.$$

- Matrix $[M_z] = \Lambda^{-k}$, where

$$\Lambda^k = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & \mathrm{Id} & 0 & \ddots \\ \ddots & 0 & \mathbf{0} & \mathrm{Id} & \ddots \\ \ddots & 0 & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Basic commuting directions: $i_k(E_\alpha)\Lambda^{kj}, j \in \mathbb{Z}, 1 \leq \alpha \leq k$.
- For LTH : $j \geq 0$ and for UTH: $j < 0$.

Matrix decompositions 1

- R be a commutative ring.
- $M_k(R)$: $k \times k$-matrices with coefficients from the ring R
- $M_{\mathbb{Z}}(R)$: $\mathbb{Z} \times \mathbb{Z}$-matrices with coefficients from R.
- To a collection of $k \times k$-matrices $\{d(ks)|s \in \mathbb{Z}\}$ in $M_k(R)$ is associated a diagonal of $k \times k$-blocks $\text{diag}(d(ks))$:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{d(kn-k)} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{d(kn)} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{d(kn+k)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Matrix decompositions 2

- The ring of $k \times k$-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \text{diag}(d(ks)) | d(ks) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$

- The elements $\Lambda^{km},\ m \in \mathbb{Z}$, act on $\mathcal{D}_k(R)$ according to the formula

$$\Lambda^{km}\mathrm{diag}(d(ks))\Lambda^{-km} = \mathrm{diag}(d(ks + km)).$$

- Each $A = (\alpha_{(i,j)}) \in M_{\mathbb{Z}}(R)$ can uniquely be written as a formal infinite sum

$$A = \sum_{j \in \mathbb{Z}} a_j \Lambda^{kj} \text{ with all the } a_j \in \mathcal{D}_k(R).$$

- Upper resp. lower triangular matrices:

$$A = \sum_{j \geq N} a_j \Lambda^{kj} \text{ resp. } B = \sum_{j \leq N} b_j \Lambda^{kj} \text{ for some } N.$$

LTH

- For $A = \sum_{j\in\mathbb{Z}} d_j \Lambda^{kj}$ one defines

$$A_{\geq 0} = \sum_{j\geq 0} d_j \Lambda^{kj}$$

- R ring of functions in the flow parameters $\{t_{i\alpha}\}$ w.r.t. $i_k(E_\alpha)\Lambda^{ki}$ and stable under all

$$\partial_{t_{i\alpha}} := \frac{\partial}{\partial t_{i\alpha}}.$$

- Deformation of Λ^k in lower triangular matrices:

$$\mathfrak{L} := \Lambda^k + \sum_{i\leq 0} m_i \Lambda^{ik}$$

- Deformation of $i_k(E_\alpha)$ in lower triangular matrices:

$$\mathfrak{U}_\alpha = i_k(E_\alpha) + \sum_{i<0} v_{i,\alpha} \Lambda^{ik}$$

The $(\Lambda^k, \mathfrak{h})$-hierarchy

- The commutatvity relations

$$[\mathcal{L}, \mathcal{U}_\alpha] = 0 \text{ and } [\mathcal{U}_\alpha, \mathcal{U}_\beta] = 0$$

- Trivially satisfied at dressing Λ^k and the $i_k(E_\beta)$:

$$\mathcal{L} = U\Lambda^k U^{-1}, \mathcal{U}_\beta = U i_k(E_\beta) U^{-1}, U = \mathrm{Id} + \sum_{i<0} u_i \Lambda^{ik}.$$

- Perturbed commuting directions: $P_{i\alpha} := \mathcal{L}^i \mathcal{U}_\alpha$.
- The Lax equations of the $(\Lambda^k, \mathfrak{h})$-hierarchy:

$$\partial_{t_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_{\geq 0}, \mathcal{L}] \text{ and } \partial_{t_{i\alpha}}(\mathcal{U}_\beta) = [(P_{i\alpha})_{\geq 0}, \mathcal{U}_\beta].$$

Zero curvature relations

- There holds:

Theorem

For deformations $\mathfrak{L}$ and the $\mathfrak{U}_\beta$ that satisfy the commutativity relations, the Lax equations are equivalent to the zero curvature equations

$$\partial_{t_{n\alpha}}(B_{m\gamma}) - \partial_{t_{m\gamma}}(B_{n\alpha}) - [B_{n\alpha}, B_{m\gamma}] = 0.$$

for the finite band matrices $B_{j\beta} = (\mathfrak{L}^j \mathfrak{U}_\beta)_{\geq 0}$.

- This set of equations expresses that the curvature of the differential form

$$\omega_{\geq 0} = \sum_{j=0}^{\infty} \sum_{\beta=1}^{k} B_{j\beta} dt_{j\beta},$$

is zero.

The relevant group

- For Hilbert spaces H_1 and H_2 and $p \geq 1$, S_p is the Schatten ideal of bounded operators $A : H_1 \mapsto H_2$

$$||A||_p^p := \mathrm{trace}(A^* A)^{\frac{p}{2}} < \infty.$$

- For each such a p one introduces the group $G(p)$ by

$$G_+ = \left\{ g = (g_{ij}) \in \mathrm{GL}(H) \;\middle|\; \begin{array}{c} \oplus_{i>j} g_{ij} \in S_p \\ \oplus_{i>j} (g^{-1})_{ij} \in S_p \end{array} \right\}.$$

- Invertible elements in the Banach algebra

$$\mathcal{G}_+ = \left\{ b = (b_{ij}) \in B(H) \;\middle|\; \oplus_{i>j} b_{ij} \in S_p \right\}$$

equiped with the norm $|| \cdot ||_{res}$ defined by

$$||b||_{res} = ||(b_{ij})||_{res} := ||b|| + || \oplus_{i>j} b_{ij}||_p.$$

Subgroups of G_+

- The Lie algebra $\mathcal{G}_+$ can be split into the sum of the Lie subalgebras

$$\mathcal{P}_+ := \left\{ p = (p_{ij}) \in \mathcal{G}_+ \;\middle|\; p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$\mathcal{U}_+ := \left\{ u = (u_{ij}) \in \mathcal{G}_+ \;\middle|\; u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

- Their corresponding Lie groups are

$$P_+ := \left\{ p = (p_{ij}) \in G_+ \;\middle|\; p_{ij} = 0 \text{ and } (p^{-1})_{ij} = 0 \text{ for all } i > j \right\},$$

$$U_+ := \left\{ u = (u_{ij}) \in G_+ \;\middle|\; \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

The big cell for LTH

- The map from $\mathcal{U}_+ \times \mathcal{P}_+$ to G_+ defined by

$$(u_+, p_+) \mapsto \exp(u_+)\exp(p_+)$$

  is a local diffeomorphism at $(0, 0)$.
- The set U_+P_+ is an open subset of G_+. It is called the *big cell* in G_+ w.r.t. U_+ and P_+.

Proposition

Let $\Omega_+ \subset G_+$ be the collection of all $g \in G_+$ such that $g_{++}(i)$ is invertible for all $i \in \mathbb{Z}$. Then Ω_+ is equal to U_+P_+.

Commuting flows 1

- Let $\mathfrak{h}$ be the subalgebra of diagonal matrices inside $M_k(\mathbb{C})$
- Let U be any open connected neighborhood of the unit circle S^1
- $\Gamma(U, \mathfrak{h})$ for the set of holomorpic maps $\gamma : U \mapsto \mathfrak{h}$ such that

$$\det(\gamma(u)) \neq 0 \text{ for all } u \in U.$$

- $\Gamma(\mathfrak{h})$ is the inductive limit of all the $\Gamma(U, \mathfrak{h})$
- Let $\Delta(\mathfrak{h})$ be the subgroup spanned by the elements

$$\begin{pmatrix} z^{m_1} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & z^{m_k} \end{pmatrix}, \text{ all } m_i \in \mathbb{Z}.$$

Commuting flows 2

Proposition

Then one has $\Gamma(\mathfrak{h}) = \Gamma_+(\mathfrak{h})\ \Delta(\mathfrak{h})\ \Gamma_-(\mathfrak{h})$, *where*

$$\Gamma_+(\mathfrak{h}) = \{\gamma_+ \mid \gamma_+ = \exp(\sum_{s\leq 0} \gamma_s z^s),\ \textit{with } \gamma_s \in \mathfrak{h} \textit{ for all } s \leq 0\}$$

and

$$\Gamma_-(\mathfrak{h}) = \{\gamma_- \mid \gamma_- = \exp(\sum_{s>0} \gamma_s z^s),\ \textit{with } \gamma_s \in \mathfrak{h} \textit{ for all } s > 0\}.$$

- The elements of $\Gamma_+(\mathfrak{h})$ give by left multiplication on H operators $M_{\gamma_+} \in P_+$. In local coordinates:

$$[M_{\gamma_+}] = \exp(\sum_{i=0}^{\infty} \sum_{\alpha=1}^{k} t_{i\alpha} i_k(E_\alpha)\Lambda^{ik}).$$

The construction

- One starts with an element $g \in G_+$. Inside the group of commuting flows $\Gamma_+(\mathfrak{h})$ one considers

$$\Gamma_+(g,\mathfrak{h}) = \{\gamma_+ \in \Gamma_+(\mathfrak{h}) \mid M_{\gamma_+} g \in \Omega_+\}.$$

- Basic result:

Proposition

The set $\Gamma_+(g,\mathfrak{h})$ is an open dense subset of $\Gamma_+(\mathfrak{h})$.

- Choose for R the ring of holomorphic functions on $\Gamma_+(g,\mathfrak{h})$
- If the element $\gamma_+ \in \Gamma_+(g,\mathfrak{h})$, there holds

$$[M_{\gamma_+}][g] = u_+(g,\gamma_+)^{-1} p_+(g,\gamma_+),$$

with $p_+(g,\gamma_+) \in [P_+]$ and $u_+(g,\gamma_+) \in [U_+]$.

Main result for LTH

- Consider $\Psi = u_+(g, \gamma_+)[M_{\gamma_+}] = \hat{\Psi}[M_{\gamma_+}]$
- Define

$$\mathcal{L}(\hat{\Psi}) = \hat{\Psi}\Lambda^k\hat{\Psi}^{-1} \text{ and } \mathcal{U}_\alpha(\hat{\Psi}) = \hat{\Psi} i_k(E_\alpha)\hat{\Psi}^{-1}.$$

- Put $P_{i\alpha} := \mathcal{L}(\hat{\Psi})^i\mathcal{U}_\alpha(\hat{\Psi})$ and $B_{i\alpha} := (P_{i\alpha})_{\geq 0}$.

Theorem

1. *For all $i \geq 0$ and all $\alpha \in \{1, \cdots, k\}$: $\partial_{t_{i\alpha}}(\Psi) = B_{i\alpha}\Psi$.*
2. *The set of matrices $(\mathcal{L}(u_+(g, \gamma_+)), \mathcal{U}_\alpha(u_+(g, \gamma_+)))$ form a solution of the $(\Lambda^k, \mathfrak{h})$-hierarchy.*
3. *For each $p_0 \in P_+$ one has*

$$\mathcal{L}(u_+(g, \gamma_+)) = \mathcal{L}(u_+(gp_0, \gamma_+)),$$
$$\mathcal{U}_\alpha(u_+(g, \gamma_+)) = \mathcal{U}_\alpha([u_+(gp_0, \gamma_+)]).$$

UTH

- For $A = \sum_{j\in\mathbb{Z}} d_j\Lambda^{kj}$ one defines

$$A_{<0} = \sum_{j<0} d_j\Lambda^{kj}$$

- R ring of functions in the flow parameters $\{s_{j\beta}\}$ w.r.t. $i_k(F_\beta)\Lambda^{-kj}$ and stable under all

$$\partial_{s_{j\beta}} := \frac{\partial}{\partial s_{j\beta}}.$$

- Deformation of Λ^{-k} in upper triangular matrices:

$$\mathcal{M} := \sum_{i\geq -1} m_i\Lambda^{ik} \text{ with } m_{-1} \text{ invertible.}$$

- Deformation of $i_k(F_\beta)$ in upper triangular matrices:

$$\mathcal{V}_\beta = \sum_{i\geq 0} v_{i,\beta}\Lambda^{ik}$$

The $(\Lambda^{-k}, \mathfrak{h})$-hierarchy

- The commutatvity relations

$$[\mathcal{M}, \mathcal{V}_\alpha] = 0 \text{ and } [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0 \tag{1}$$

- Trivially satisfied at dressing Λ^{-k} and the $i_k(F_\beta)$:

$$\mathcal{M} = V\Lambda^{-k}V^{-1}, \mathcal{V}_\beta = Vi_k(F_\beta)V^{-1}, V = \sum_{i \geq 0} v_i \Lambda^{ik}.$$

- Perturbed commuting directions: $Q_{j\alpha} := \mathcal{M}^j \mathcal{V}_\alpha$.
- The Lax equations of the $(\Lambda^{-k}, \mathfrak{h})$-hierarchy:

$$\partial_{s_{j\alpha}}(\mathcal{M}) = [(Q_{j\alpha})_{<0}, \mathcal{M}] \text{ and } \partial_{s_{j\alpha}}(\mathcal{V}_\beta) = [(Q_{j\alpha})_{<0}, \mathcal{V}_\beta].$$

The relevant geometry for UTH

- For each p corresponding to the Schatten class S_p the group G_- is

$$G_- = \left\{ g = (g_{ij}) \in \mathrm{GL}(H) \;\middle|\; \begin{array}{c} \oplus_{i<j} g_{ij} \in S_p \\ \oplus_{i<j} (g^{-1})_{ij} \in S_p \end{array} \right\}.$$

- Its Lie algebra is

$$\mathcal{G}_- = \left\{ b = (b_{ij}) \in B(H) \;\middle|\; \oplus_{i<j} b_{ij} \in S_p \right\}$$

- It splits into the sum of the Lie subalgebras

$$\mathcal{P}_- := \left\{ p = (p_{ij}) \in \mathcal{G}_- \;\middle|\; p_{ij} = 0 \text{ for all } i > j \right\},$$

$$\mathcal{U}_- := \left\{ u = (u_{ij}) \in \mathcal{G}_- \;\middle|\; u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

More geometry

- Their corresponding Lie groups are

$$P_- := \left\{ p = (p_{ij}) \in G_- \;\middle|\; p_{ij} = 0 \text{ and } (p^{-1})_{ij} = 0 \text{ for all } i > j \right\},$$

$$U_- := \left\{ u = (u_{ij}) \in G_- \;\middle|\; \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

- The set $\Omega_- = U_- P_-$ is an open subset of G_-. It is called the *big cell* in G_- w.r.t. U_- and P_-.
- The elements of $\Gamma_-(\mathfrak{h})$ give by left multiplication on H operators $M_{\gamma_-} \in U_-$. In local coordinates:

$$[M_{\gamma_-}] = \exp\left(\sum_{j=1}^{\infty} \sum_{\beta=1}^{k} s_{j\beta} i_k(F_\beta) \Lambda^{-jk}\right).$$

The construction for UTH

- One starts with an element $g \in G_-$. Inside the group of commuting flows $\Gamma_-(\mathfrak{h})$ one considers

$$\Gamma_-(g,\mathfrak{h}) = \{\gamma_- \in \Gamma_-(\mathfrak{h}) \mid gM_{\gamma_-}^{-1} \in \Omega_-\}.$$

- Basic result:

Proposition

The set $\Gamma_-(g,\mathfrak{h})$ is an open dense subset of $\Gamma_-(\mathfrak{h})$.

- Choose for R the ring of holomorphic functions on $\Gamma_-(g,\mathfrak{h})$
- If the element $\gamma_- \in \Gamma_-(g,\mathfrak{h})$, there holds

$$g[M_{\gamma_-}^{-1}] = u_-(g,\gamma_-)p(g,\gamma_-),$$

with $p(g,\gamma_-) \in P_-$ and $u_-(g,\gamma_-) \in U_-$.

Main result for UTH

- Consider $\Phi = [p(g,\gamma_-)][M_{\gamma_-}] = \hat{\Phi}[M_{\gamma_-}]$
- Define

$$\mathcal{M}(\hat{\Phi}) = \hat{\Phi}\Lambda^{-k}\hat{\Phi}^{-1} \text{ and } \mathcal{V}_\beta(\hat{\Phi}) = \hat{\Phi} i_k(F_\beta)\hat{\Phi}^{-1}.$$

- Put $Q_{j\beta} := \mathcal{M}(\hat{\Phi})^j \mathcal{V}_\beta(\hat{\Phi})$ and $C_{j\beta} := (Q_{j\beta})_{<0}$.

Theorem

1. *For all* $j \geq 1$ *and all* $\beta \in \{1, \cdots, k\}$*:* $\partial_{s_{j\beta}}(\Phi) = C_{j\beta}\Phi$.
2. *The set of matrices* $(\mathcal{M}([p(g,\gamma_-,)]), \mathcal{V}_\beta([p(g,\gamma_-)]))$ *form a solution of the* $(\Lambda^{-k}, \mathfrak{h})$*-hierarchy.*
3. *For each* $u_0 \in U_-$ *one has*

$$\mathcal{M}([p(g,\gamma_-)]) = \mathcal{M}([p(u_0 g,\gamma_-)]),$$
$$\mathcal{V}_\beta([p(g,\gamma_-)]) = \mathcal{V}_\beta([p(u_0 g,\gamma_-)]).$$

LTH+UTH

- R ring of functions in the flow parameters $\{t_{i\alpha}\}$ and $\{s_{j\beta}\}$ w.r.t. the $i_k(E_\alpha)\Lambda^{ki}$ and the $i_k(F_\beta)\Lambda^{-kj}$ and stable under all

$$\partial_{t_{i\alpha}} := \frac{\partial}{\partial t_{i\alpha}} \text{ and } \partial_{s_{j\beta}} := \frac{\partial}{\partial s_{j\beta}}.$$

- Deformations $(\mathcal{L}, \mathcal{U}_\alpha)$ of Λ^k and the $i_k(E_\alpha)$ in lower triangular matrices.
- These directions should commute:

$$[\mathcal{L}, \mathcal{U}_\alpha] = 0 \text{ and } [\mathcal{U}_\alpha, \mathcal{U}_\beta] = 0$$

- Deformations $(\mathcal{M}, \mathcal{V}_\beta)$ of Λ^{-k} and the $i_k(F_\beta)$ in upper triangular matrices:
- These directions should commute:

$$[\mathcal{M}, \mathcal{V}_\alpha] = 0 \text{ and } [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0$$

The two dimensional $\mathfrak{h}$-hierarchy

- The Lax equations for the two-dimensional $\mathfrak{h}$-hierarchy are:
- For $\mathcal{L}$ and the $\mathcal{U}_\alpha$:

$$\partial_{t_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_{\geq 0}, \mathcal{L}] \text{ and } \partial_{t_{i\alpha}}(\mathcal{U}_\beta) = [(P_{i\alpha})_{\geq 0}, \mathcal{U}_\beta],$$

$$\partial_{s_{j\gamma}}(\mathcal{L}) = [(Q_{j\gamma})_{<0}, \mathcal{L}] \text{ and } \partial_{s_{j\gamma}}(\mathcal{U}_\beta) = [(Q_{j\gamma})_{<0}, \mathcal{U}_\beta].$$

- For $\mathcal{M}$ and the $\mathcal{V}_\beta$:

$$\partial_{s_{j\gamma}}(\mathcal{M}) = [(Q_{j\gamma})_{<0}, \mathcal{M}] \text{ and } \partial_{s_{j\gamma}}(\mathcal{V}_\beta) = [(Q_{j\gamma})_{<0}, \mathcal{V}_\beta],$$

$$\partial_{t_{i\alpha}}(\mathcal{M}) = [(P_{i\alpha})_{\geq 0}, \mathcal{M}] \text{ and } \partial_{t_{i\alpha}}(\mathcal{V}_\beta) = [(P_{i\alpha})_{\geq 0}, \mathcal{V}_\beta].$$

The group setting

- Let G be the group $\{g \in \mathrm{GL}(H) \mid g - \mathrm{Id} \in S_p\}$.
- Note $\Gamma_+(\mathfrak{h}) \not\subseteq G$ and $\Gamma_-(\mathfrak{h}) \not\subseteq G$.
- The group $U_+ \subset G$. Let $P = P_- \cap G$.
- Big cell in G: $U_+ P$.

Proposition

For each $g \in G$, there is a $\gamma_+ \in \Gamma_+(\mathfrak{h})$ and a $\gamma_- \in \Gamma_-(\mathfrak{h})$ such that

$$M_{\gamma_+} M_{\gamma_-} g M_{\gamma_+}^{-1} M_{\gamma_-}^{-1}$$

belongs to the big cell U_+P. The collection of all these $(\gamma_+, \gamma_-) \in \Gamma_+(\mathfrak{h}) \times \Gamma_-(\mathfrak{h})$ one denotes by $\Gamma(g, \mathfrak{h})$

Final result

- For $g \in G$ and $(\gamma_+, \gamma_-) \in \Gamma(g, \mathfrak{h})$

$$[M_{\gamma_+}][M_{\gamma_-}][g][M_{\gamma_+}^{-1}][M_{\gamma_-}^{-1}] = \hat{\Psi}^{-1}\hat{\Phi},$$

with $\hat{\Psi} \in [U_+]$ and $\hat{\Phi} \in [P]$.

- Write

$$\mathcal{L}(\hat{\Psi}) = \hat{\Psi}\Lambda^k\hat{\Psi}^{-1} \text{ and } \mathcal{U}_\alpha(\hat{\Psi}) = \hat{\Psi}i_k(E_\alpha)\hat{\Psi}^{-1}$$

- and

$$\mathcal{M}(\hat{\Phi}) = \hat{\Phi}\Lambda^{-k}\hat{\Phi}^{-1} \text{ and } \mathcal{V}_\beta(\hat{\Phi}) = \hat{\Phi}i_k(F_\beta)\hat{\Phi}^{-1}.$$

- There holds now

Theorem

The matrices $(\mathcal{L}(\hat{\Psi}), \mathcal{U}_\alpha(\hat{\Psi}))$ and $(\mathcal{M}(\hat{\Phi}), \mathcal{V}_\beta(\hat{\Phi}))$ satisfy the Lax equations of the two-dimensional $\mathfrak{h}$-hierarchy.

THANK YOU FOR YOUR ATTENTION